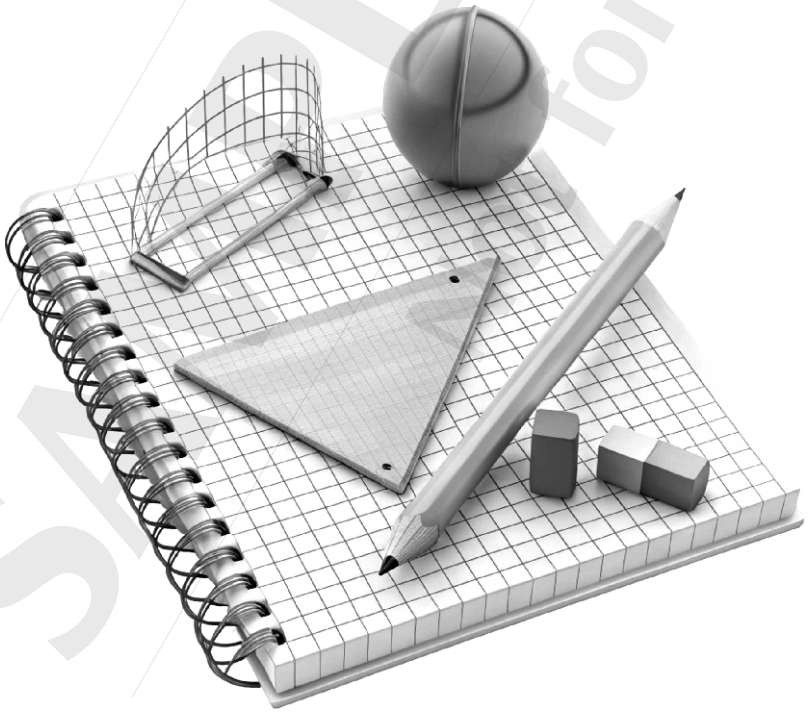
SLO-Based

Class 12th

Unique Notes

MATHEMATICS

Prepared for the Session 2026-27
According to the New Curriculum of PECTAA 2025-26



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UNIQUE NOTES, MATHEMATICS 12th

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Chapter 02 FURTHER DIFFERENTIATION

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Differentiation, a fundamental concept in calculus introduced by Newton and Leibniz, allows us to study how quantities change and evolve.

Review of Differentiation Basics

Definition: The derivative of the function $y = f(x)$ is denoted by $\frac{dy}{dx}$ or $f'(x)$ and is defined as:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$ if exists or

$$\frac{dy}{dx} = \lim_{\delta\to 0}\frac{\delta y}{\delta x}$$ if exists.

The derivative gives the instantaneous rate of change of the function at a point.

Recall?

- **Power Rule:** $\frac{d}{dx}(x^n) = n.x^{n-1}$ where $n \in \mathbb{R}$.
- **Constant Rule:** $\frac{d}{dx}(c) = 0$, for any $c \in \mathbb{R}$.

In the following rules $u = u(x), v = v(x)$ are differentiable functions and c is any arbitrary constant.

- **Constant Multiple Rule:** $\frac{d}{dx}(cu) = c\frac{du}{dx} = cu'.$
- **Sum and Difference Rule:** $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx} = u' \pm v'$
- **Product Rule:** This rule helps to differentiable the product of two functions.

$$\frac{d}{dx}(uv) = u'v + uv'$$

- **Quotient Rule:** This rule is used to differentiate the division of two functions.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

Where $v(x) \neq 0$ for all x in the domain.

Derivatives of Trigonometric Functions

Derivative of the Sine Function

While finding derivatives of trigonometric functions, we assume that the angle of measured in radians. To calculate the derivative of $f(x) = \sin x.$

Here, $f(x+h) = \sin(x+h)$, and

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

(By the definition of derivative)

$$= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

(By substituting the values)

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$$= \lim_{h\to 0} \frac{2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)}{h}$$

$$\because \sin p - \sin q = 2\cos\frac{p+q}{2}\sin\frac{p-q}{2}$$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\cdot \lim_{h\to 0}\frac{\sin(h/2)}{h/2}$$ applying limit $h \to 0$

$$= \cos\left(\frac{2x}{2}+0\right)(1)$$

$$= \cos x(1) = \cos x$$

$\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$, θ in radians.

Thus $\frac{d}{dx}(\sin x) = \cos x$

The derivative of the sine function is the cosine function $\frac{d}{dx}(\sin x) = \cos x$

Derivative of the Other Trigonometric Functions

After finding derivatives of $\sin x$ and $\cos x$ by definition, the derivatives of the remaining trigonometric functions can be found using product rule and quotient rule of differentiation.

The Derivatives of Six Trigonometric Functions

(i) $\frac{d}{dx}(\sin x) = \cos x$

(ii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(iii) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(iv) $\frac{d}{dx}(\cos x) = -\sin x$

(v) $\frac{d}{dx}(\cot x) = -\csc^2 x$

(vi) $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Example 1: Find $f'(x)$ if $f(x) = \tan x$ (By using definition) 12302001

Solution: $f(x) = \tan x = \frac{\sin x}{\cos x}$

Differentiating w.r.t. 'x', we have

$$f'(x) = \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x}$$

$$\because \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$= \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$\because (\sin x)' = \cos x, (\cos x)' = -\sin x$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x \qquad \because \sin^2 x + \cos^2 x = 1$$

Thus, $\frac{d}{dx}(\tan x) = \sec^2 x.$

Example 2: Find $\frac{dy}{dx}$ if $y = x^2\csc x$ 12302002

Solution: $y = x^2\csc x$

Differentiating w.r.t. 'x'

$$\frac{dy}{dx} = (x^2)'\csc x + x^2(\csc x)'$$

$$= 2x\csc x + x^2(-\csc x\cdot\cot x)$$

$$= 2x\csc x - x^2\csc x\cot x$$

$$= x\csc x(2 - x\cot x)$$

The Chain Rule

Composition of Functions

Let $X \to Y$, and $Y \to Z$, then their composition $fog : X \to Y$ is defined as:

$$fog(x) : f(g(x))$$

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For example, consider two real valued functions $g(x)=x^2-x$ and $f(x)=\sin x$ then

$$f\circ g(x)=f\left[g(x)\right]=f\left(x^2-x\right)=\sin\left(x^2-x\right)$$

Theorem 1: (The Chain Rule) **12302003**

If $f(u)$ is differentiable at the point $u=g(x)$ and $g(x)$ is differentiable at x, then the composite function $f\circ g(x)=f\left(g(x)\right)$ is differentiable at x, an

$$f\circ g'(x)=f'\left(g(x)\right)\cdot g'(x)$$

In Leibniz's notation, if $y=f(u)$ and $u=g(x)$, then

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

Example 3: Differentiate $y=\sin\left(x^2-x\right)$ w.r.t 'x'. **12302004**

Solution: Let $u=x^2-x$ .....(i)

Substituting $u=x^2-x$ into $y=\sin\left(x^2-x\right)$, we have $y=\sin u$.

Now, we have $y=\sin u$ and $u=x^2-x$

Differentiate with respect to u.

$$\frac{d}{du}(y)=\frac{d}{du}(\sin u)$$

$$\frac{dy}{du}=\cos u$$

Differentiate with respect to x.

$$\frac{d}{dx}(u)=\frac{d}{dx}\left(x^2-x\right)$$

$$\frac{du}{dx}=2x-1$$

By Chain Rule

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{dy}{dx}=\cos u(2x-1)$$

$$\boxed{\frac{dy}{dx}=\cos\left(x^2-x\right)(2x-1)} \qquad \because u=x^2-x$$

[By using equation (i)]

Derivative of a Function Given in the Form of Parametric Equations

Let a function be represented by parametric equations

$$x=f(t) \quad \text{and} \quad y=g(t)$$

From $x=f(t)$, we have

$t=f^{-1}(x)$ (Provided f is bijective)

Now, by Chain Rule applying to $y=g(t)$ and $t=f^{-1}(x)$, we get:

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \boxed{\because \frac{dt}{dx}=\frac{1}{\frac{dx}{dt}}}$$

Example 4: Find $\frac{dy}{dx}$ if $y=a\cos t$ and $x=a\sin t$. **12302005**

Solution: Differentiate $y=a\cos t$ and $x=a\sin t$ w.r.t 't'

$$\frac{dy}{dt}=-a\sin t \quad \text{and} \quad \frac{dx}{dt}=a\cos t$$

Apply the Chain Rule

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=-\frac{a\sin t}{a\cos t}=-\tan t$$

Implicit Differentiation

When an implicit relation cannot be easily solved for y (or when multiple explicit functions are involved), we can still compute $\frac{dy}{dx}$ using implicit differentiation. The steps are as follows:

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1. Differentiate both sides of $F(x,y)=0$ with respect to x, treating y as an implicit function of x.
2. Apply the Chain Rule to terms containing y.
3. Solve the resulting equation for $\frac{dy}{dx}$.

Example 5: Find $\frac{dy}{dx}$ if $x+y^2+\sin(xy^2)=0$

12302006

Solution: Differentiate $x+y^2+\sin(xy^2)=0$ with respect to x to obtain:

$$\frac{d}{dx}\left[x+y^2+\sin(xy^2)\right]=0$$

$$\frac{d}{dx}(x)+\frac{d}{dx}(y^2)+\frac{d}{dx}\left(\sin(xy^2)\right)=0$$

$$1+2y\frac{dy}{dx}+\cos(xy^2)\cdot\frac{d}{dx}(xy^2)=0$$

$$2y\frac{dy}{dx}+\cos(xy^2)\left\{y^2+x\left(2y\frac{dy}{dx}\right)\right\}=-1$$

$$2y\frac{dy}{dx}+y^2\cos(xy^2)+2xy\frac{dy}{dx}\cos(xy^2)=-1$$

$$2y\frac{dy}{dx}\left\{1+x\cos(xy^2)\right\}=-1-y^2\cos(xy^2)$$

$$\boxed{\frac{dy}{dx}=-\frac{1+y^2\cos(xy^2)}{2y\left\{1+x\cos(xy^2)\right\}}}$$

Derivatives of Inverse Trigonometric Functions

Here, we will prove that:

(i) $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$, for $x\in(-1,1)$

(ii) $\frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}}$, for $x\in(-1,1)$

(iii) $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$, for $x\in\mathbb{R}$

(iv) $\frac{d}{dx}(\cot^{-1}x)=-\frac{1}{1+x^2}$, for $x\in\mathbb{R}$

(v) $\frac{d}{dx}(\sec^{-1}x)=-\frac{1}{|x|\sqrt{x^2-1}}$, for $x\in(-\infty,-1)\cup(1,\infty)$

(vi) $\frac{d}{dx}(\csc^{-1}x)=-\frac{1}{|x|\sqrt{x^2-1}}$, for $x\in(-\infty,-1)\cup(1,\infty)$

Proof (1): Let $y=\sin^{-1}(x)$, where $x\in[-1,1]$ and $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

$$\sin y=x \qquad \ldots\ldots\ldots(i)$$

Differentiate equation (i) w.r.t. x, we have

$$\cos y\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{\cos^2 y}}$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-\sin^2 y}}$$

$$\boxed{\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}},\ for\ x\in(-1,1)\ \ (\because \sin y=x)}$$

Proof (2): Let $y=\cot^{-1}(x)$, where $y\in(0,\pi)$ and $x\in\mathbb{R}$.

$$\cot y=x \qquad \ldots\ldots\ldots(ii)$$

We differentiate equation (i) w.r.t. x,

$$\frac{d}{dx}(\cot y)=\frac{d}{dx}(x)$$

$$-\csc^2 y\frac{dy}{dx}=1 \qquad \left((\cot y)'=-\csc^2 y\right)$$

$$\frac{dy}{dx}=-\frac{1}{\csc^2 y} \qquad \left(\csc y\neq 0,\ for\ y\in(0,\pi)\right)$$

$$=-\frac{1}{1+\cot^2 y} \qquad \left(\because 1+\cot^2 y=\csc^2 y\right)$$

$$\boxed{\frac{d}{dx}(\cot^{-1}x)=\frac{-1}{1+x^2}}\ \text{for}\ x\in R\ (\because \cot y=x)$$

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Proof (3): Let $y = \sec^{-1} x$, where $x \in (-\infty,-1] \cup [1,\infty)$ and $y \in [0,\pi] - \left\{\frac{\pi}{2}\right\}$

$\sec y = x$ .....(i) , $|x| \geq 1$

We differentiate equation (i) w.r.t. x:

$$\frac{d}{dx}(\sec y) = \frac{d}{dx}(x)$$

$$\sec y \tan y \frac{dy}{dx} = 1 \qquad \left[\because (\sec y)' = \sec y \tan y\right]$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$ where, $\sec y \neq 0$ and $\tan y \neq 0$

$\tan y \neq 0 \Rightarrow y \neq 0$ and $y \neq \pi \Rightarrow x \neq -1,1$ and from equation (i)

$$\frac{dy}{dx} = \frac{1}{x \tan y} \text{.....(ii)} \qquad \boxed{\because \sec y = x}$$

$$\tan y = \begin{cases} \sqrt{\tan^2 y}, & y \in \left(0,\frac{\pi}{2}\right) \\ -\sqrt{\tan^2 y}, & y \in \left(\frac{\pi}{2},\pi\right) \end{cases}$$

$$\tan y = \begin{cases} \sqrt{\sec^2 y - 1}, & y \in \left(0,\frac{\pi}{2}\right) \\ -\sqrt{\sec^2 y - 1}, & y \in \left(\frac{\pi}{2},\pi\right) \end{cases}$$

$$\tan y = \begin{cases} \sqrt{x^2-1}, & x \in (1,\infty) \\ -\sqrt{x^2-1}, & x \in (-\infty,-1) \end{cases}$$

$$\tan y = \begin{cases} \sqrt{x^2-1}, & x \in (1,-1) \\ -\sqrt{x^2-1}, & x \in (-\infty,1) \end{cases}$$

$\because \sec y = x$, *and* $x \in (1,\infty)$ *when* $y \in \left(0,\frac{\pi}{2}\right)$, *while* $x \in (-\infty,-1)$ *when* $y \in \left(\frac{\pi}{2},\pi\right)$

Substitute the value of $\tan y$ in equation (ii)

$$\frac{d}{dx} = \begin{cases} \frac{1}{x\sqrt{x^2-1}}, & x \in (1,\infty) \\ \frac{1}{-x\sqrt{x^2-1}}, & x \in (-\infty,-1) \end{cases}$$

Hence, $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$, where $x \in (-\infty,-1) \cup (1,\infty)$ $\left(\because |x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}\right)$

Example 6: Find $\frac{dy}{dx}$ if

$y = x\cos^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2-x^2}$. **12302007**

Solution: Differentiate $y = x\cos^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2-x^2}$ w.r.t. x to obtain:

$$\frac{d}{dx}(y) = \frac{d}{dx}\left[x\cos^{-1}\left(\frac{x}{a}\right) - \sqrt{a^2-x^2}\right]$$

$$\frac{dy}{dx} = \cos^{-1}\left(\frac{x}{a}\right)\frac{d}{dx}(x) + x\frac{d}{dx}\cos^{-1}\left(\frac{x}{a}\right) - \frac{d}{dx}\left(\sqrt{a^2-x^2}\right)$$

$$\frac{dy}{dx} = \cos^{-1}\left(\frac{x}{a}\right) + x\left(-\frac{1}{\sqrt{1-(x/a)^2}}\cdot\frac{d}{dx}\left(\frac{x}{a}\right)\right) - \frac{1}{2}\left(a^2-x^2\right)^{-1/2}\cdot\frac{d}{dx}\left(a^2-x^2\right)$$

$$\frac{dy}{dx} = \cos^{-1}\left(\frac{x}{a}\right) - \frac{x}{\sqrt{\frac{a^2-x^2}{a^2}}}\cdot\frac{1}{a} - \frac{1}{2}\cdot\frac{1}{\sqrt{a^2-x^2}}(-2x)$$

$$\frac{dy}{dx} = \cos^{-1}\left(\frac{x}{a}\right) - \frac{x}{\sqrt{a^2-x^2}} + \frac{x}{\sqrt{a^2-x^2}}$$

$$= \cos^{-1}\left(\frac{x}{a}\right)$$

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EXERCISE 2.1

1. Derive the formula for the derivative of the following function w.r.t. x:

(a) $\sec x$ **12302008**

Solution:

Let $y = \sec x$

$$y = \frac{1}{\cos x}$$

Differentiating w.r.t. 'x':

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$\frac{dy}{dx} = \frac{(\cos x)\frac{d}{dx}(1) - (1)\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$\because \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{(v)^2}$$

$$= \frac{(\cos x)(0) - (1)(-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x \cdot \cos x}$$

$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$

$$\frac{dy}{dx} = \sec x \tan x$$

(b) $\csc x$ **12302009**

Solution:

Let $y = \operatorname{cosec} x$

$$y = \frac{1}{\sin x}$$

Differentiating w.r.t. 'x':

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

$$\frac{dy}{dx} = \frac{(\sin x)\frac{d}{dx}(1) - (1)\frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{(\sin x)(0) - 1(\cos x)}{\sin^2 x}$$

$$= \frac{0 - \cos x}{\sin^2 x}$$

$$= \frac{-\cos x}{\sin^2 x}$$

$$= -\frac{\cos x}{\sin x \cdot \sin x}$$

$$= \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$$\frac{dy}{dx} = -\operatorname{cosec} x \cot x$$

(c) $\cot x$ **12302010**

Solution:

Let $y = \cot x$

$$y = \frac{\cos x}{\sin x}$$

Differentiating w.r.t. 'x':

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$

$$\frac{dy}{dx} = \frac{(\sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= -\frac{(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$\because \sin^2 x + \cos^2 x = 1$$

$$= \frac{-1}{\sin^2 x}$$

$$\frac{dy}{dx} = -\operatorname{cosec}^2 x$$

In 2 – 6, differentiate with respect to the variable involved.

2. $y = x^2 \cos x$ **12302011**

Solution:

$$y = x^2 \cos x$$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^2 \cos x)$$

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$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\cos x) + (\cos x)\frac{d}{dx}(x^2)$$

$$\boxed{\because \frac{d}{dx}(uv) = uv' + vu'}$$

$$= x^2(-\sin x) + (\cos x)(2x)$$

$$= -x^2 \sin x + 2x\cos x$$

$$= x(-x\sin x + 2\cos x)$$

$$= x(2\cos x - x\sin x)$$

3. $y = x\sec x + \frac{1}{x}$ **12302012**

Solution:

$$y = x\sec x + \frac{1}{x}$$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(x\sec x + \frac{1}{x}\right)$$

$$= \frac{d}{dx}(x\sec x) + \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= x\frac{d}{dx}(\sec x) + (\sec x)\frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= x(\sec x\tan x) + (\sec x)(1) + (-1)(x)^{-2}$$

$$= x\sec x\tan x + \sec x - \frac{1}{(x)^2}$$

$$\frac{dy}{dx} = \sec x(x\tan x + 1) - \frac{1}{x^2}$$

4. $y = \frac{\sin t}{1-\cos t}$ **12302013**

Solution:

$$y = \frac{\sin t}{1-\cos t}$$

Differentiate w.r.t. 't':

$$\frac{d}{dt}(y) = \frac{d}{dt}\left(\frac{\sin t}{1-\cos t}\right)$$

$$\frac{dy}{dt} = \frac{(1-\cos t)\frac{d}{dt}(\sin t) - (\sin t)\frac{d}{dt}(1-\cos t)}{(1-\cos t)^2}$$

$$\boxed{\because \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}}$$

$$= \frac{(1-\cos t)(\cos t) - (\sin t)[0-(-\sin t)]}{(1-\cos t)^2}$$

$$= \frac{\cos t - \cos^2 t - \sin^2 t}{(1-\cos t)^2}$$

$$= \frac{\cos t - (\sin^2 t + \cos^2 t)}{(1-\cos t)^2}$$

$$\boxed{\because \sin^2 t + \cos^2 t = 1}$$

$$= \frac{\cos t - 1}{(1-\cos t)^2}$$

$$= -\frac{(1-\cos t)}{(1-\cos t)^2}$$

$$\frac{dy}{dt} = -\frac{1}{(1-\cos t)}$$

5. $y = (1+\csc t)\sin t$ **12302014**

Solution:

$$y = (1+\csc t)\sin t$$

$$y = \sin t + \operatorname{cosec} t \sin t$$

$$y = \sin t + \frac{1}{\sin t}\cdot \sin t$$

$$y = \sin t + 1$$

Differentiate w.r.t. 't':

$$\frac{d}{dt}(y) = \frac{d}{dt}(\sin t + 1)$$

$$\frac{dy}{dt} = \frac{d}{dt}(\sin t) + \frac{d}{dt}(1)$$

$$\frac{dy}{dt} = \cos t + 0$$

$$\frac{dy}{dt} = \cos t$$

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6. $y = \sec x + 4\sqrt{x} - 10$ **12302015**

Solution:

$y = \sec x + 4\sqrt{x} - 10$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\sec x + 4\sqrt{x} - 10\right)$$

$$\frac{dy}{dx} = \frac{d}{dx}(\sec x) + 4\frac{d}{dx}\sqrt{x} - \frac{d}{dx}(10)$$

$$= (\sec x \tan x) + 4\cdot\left(\frac{1}{2}(x)^{\frac{1}{2}-1}\right) - 0$$

$$= (\sec x \tan x) + 2(x)^{-1/2}$$

$$\frac{dy}{dx} = \sec x \tan x + \frac{2}{\sqrt{x}}$$

In 7 – 8, find $\frac{dy}{dx}$ by making suitable substitutions and applying the Chain Rule.

7. $y = \sqrt{\frac{1+x}{1-x}}$ **12302016**

Solution:

$y = \sqrt{\frac{1+x}{1-x}}$

Let $u = \frac{1+x}{1-x}$ so, $y = (u)^{1/2}$

Now, $y = (u)^{1/2}$

Differentiate w.r.t. 'u':

$$\frac{dy}{du} = \frac{d}{du}(u)^{1/2}$$

$$\frac{dy}{du} = \frac{1}{2}(u)^{-1/2}$$

$$= \frac{1}{2\sqrt{u}}$$

Now, $u = \frac{1+x}{1-x}$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(u) = \frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

$$\frac{du}{dx} = \frac{(1-x)\frac{d}{dx}(1+x) - (1+x)\frac{d}{dx}(1-x)}{(1-x)^2}$$

$$\frac{du}{dx} = \frac{(1-x)(0+1) - (1+x)(0-1)}{(1-x)^2}$$

$$\frac{du}{dx} = \frac{(1-x)(1) - (1+x)(-1)}{(1-x)^2}$$

$$\frac{du}{dx} = \frac{1-x+1+x}{(1-x)^2}$$

$$\frac{du}{dx} = \frac{2}{(1-x)^2}$$

Now, By Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}}\cdot\frac{2}{(1-x)^2}$$

$$= \frac{1}{\sqrt{u}(1-x)^2}$$

Putting the value of u, we have

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{1+x}{1-x}}(1-x)^2}$$

$$= \frac{1}{\frac{\sqrt{1+x}}{\sqrt{1-x}}\cdot(1-x)^2}$$

$$= \frac{1}{\sqrt{1+x}(1-x)^{-1/2}\cdot(1-x)^2}$$

$$= \frac{1}{\sqrt{1+x}\cdot(1-x)^{\frac{-1}{2}+2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x}(1-x)^{3/2}}$$

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8. $y=(\sin x+\cos x)^{3/2}$ **12302017**

Solution:

$y=(\sin x+\cos x)^{3/2}$

Let $u=\sin x+\cos x$, So, $y=(u)^{\frac{3}{2}}$

So, $y=(u)^{3/2}$

Differentiate w.r.t. 'u':

$$\frac{d}{du}(y)=\frac{d}{du}(u^{3/2})$$
$$\frac{dy}{du}=\frac{3}{2}u^{\frac{3}{2}-1}$$
$$\frac{dy}{du}=\frac{3}{2}(u)^{1/2}$$

Now, $u=\sin x+\cos x$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(u)=\frac{d}{dx}(\sin x+\cos x)$$
$$\frac{du}{dx}=\frac{d}{dx}(\sin x)+\frac{d}{dx}(\cos x)$$
$$\frac{du}{dx}=\cos x-\sin x$$

By Chain Rule

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$
$$=\frac{3}{2}(u)^{1/2}\cdot(\cos x-\sin x)$$

Putting the value of 'u', we have:

$$\frac{dy}{dx}=\frac{3}{2}(\sin x+\cos x)^{1/2}(\cos x-\sin x)$$
$$\frac{dy}{dx}=\frac{3}{2}\sqrt{\sin x+\cos x}(\cos x-\sin x)$$

In 9 – 12, differentiate w.r.t. the variable involved.

9. $y=\sec(\tan x)$ **12302018**

Solution:

$y=\sec(\tan x)$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y)=\frac{d}{dx}[\sec(\tan x)]$$
$$\frac{dy}{dx}=[\sec(\tan x)\tan(\tan x)]\frac{d}{dx}(\tan x)$$
$$\frac{dy}{dx}=[\sec(\tan x)\tan(\tan x)]\sec^2 x$$

10. $y=\sqrt{\sin x}+\cos\sqrt{x}$ **12302019**

Solution:

$y=\sqrt{\sin x}+\cos\sqrt{x}$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y)=\frac{d}{dx}(\sqrt{\sin x}+\cos\sqrt{x})$$
$$\frac{dy}{dx}=\frac{d}{dx}\sqrt{\sin x}+\frac{d}{dx}(\cos\sqrt{x})$$
$$\frac{dy}{dx}=\frac{d}{dx}(\sin x)^{1/2}+\frac{d}{dx}(\cos\sqrt{x})$$
$$\frac{dy}{dx}=\frac{1}{2}(\sin x)^{-1/2}\frac{d}{dx}(\sin x)+(-\sin\sqrt{x})\frac{d}{dx}(\sqrt{x})$$
$$\frac{dy}{dx}=\frac{1}{2\sqrt{\sin x}}(\cos x)-(\sin\sqrt{x})\left(\frac{1}{2}(x)^{-1/2}\right)$$
$$\frac{dy}{dx}=\frac{\cos x}{2\sqrt{\sin x}}-\frac{\sin\sqrt{x}}{2\sqrt{x}}$$

11. $y=\tan^2 x+\cot(x^2)$ **12302020**

Solution:

$y=\tan^2 x+\cot(x^2)$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y)=\frac{d}{dx}[\tan^2 x+\cot(x^2)]$$
$$\frac{dy}{dx}=\frac{d}{dx}(\tan^2 x)+\frac{d}{dx}\cot(x^2)$$
$$\frac{dy}{dx}=2(\tan x)\frac{d}{dx}(\tan x)+[-\operatorname{cosec}^2(x^2)]\frac{d}{dx}(x^2)$$
$$\frac{dy}{dx}=2(\tan x)(\sec^2 x)-\operatorname{cosec}^2(x^2)(2x)$$
$$\frac{dy}{dx}=2\tan x\sec^2 x-2x\operatorname{cosec}^2(x^2)$$

12. $y=\sin^3(\cos 2v)$ **12302021**

Solution:

$y=\sin^3(\cos 2v)$

$y=[\sin(\cos 2v)]^3$

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Differentiate w.r.t. 'v':

$$\frac{d}{dv}(y)=\frac{d}{dv}\left[\sin\left(\cos 2v\right)\right]^3$$

$$\frac{dy}{dv}=3\left[\sin\left(\cos 2v\right)\right]^2\frac{d}{dv}\left[\sin\left(\cos 2v\right)\right]$$

$$=3\left[\sin^2\left(\cos 2v\right)\right]\left[\cos\left(\cos 2v\right)\right]\frac{d}{dv}\left(\cos 2v\right)$$

$$=3\left[\sin^2\left(\cos 2v\right)\right]\left[\cos\left(\cos 2v\right)\right]\left(-\sin 2v\right)\frac{d}{dv}\left(2v\right)$$

$$=3\left[\sin^2\left(\cos 2v\right)\right]\left[\cos\left(\cos 2v\right)\right]\left(-\sin 2v\right)\left(2\right)$$

$$\frac{dy}{dv}=-6\sin 2v\cdot\cos\left(\cos 2v\right)\cdot\sin^2\left(\cos 2v\right)$$

13. Find $\frac{dy}{dx}$ for the following parametric equations:

(i) $x=\sin^2 t$ **and** $y=\cos 2t$ **12302022**

Solution:

$x=\sin^2 t$ .....(i)

$y=\cos 2t$ .....(ii)

From equation (i)

$$x=\left(\sin t\right)^2$$

Differentiate w.r.t. 't':

$$\frac{d}{dt}(x)=\frac{d}{dt}\left(\sin t\right)^2$$

$$\frac{dx}{dt}=2\sin t\cdot\frac{d}{dt}\left(\sin t\right)$$

$$\frac{dx}{dt}=2\sin t\cos t$$

$$\Rightarrow\frac{dt}{dx}=\frac{1}{2\sin t\cos t}$$

From equation (ii),

$$y=\cos 2t$$

Differentiate w.r.t. 't':

$$\frac{d}{dt}(y)=\frac{d}{dt}\left(\cos 2t\right)$$

$$\frac{dy}{dt}=\left(-\sin 2t\right)\frac{d}{dt}\left(2t\right)$$

$$\frac{dy}{dt}=-\sin 2t\left(2\right)$$

$$\frac{dy}{dt}=-2\sin 2t$$

By Chain Rule,

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$=\frac{-2\sin\left(2t\right)}{2\sin t\cos t}$$

$$=\frac{-2\left(2\sin t\cos t\right)}{2\sin t\cos t}$$

$$\because\ \sin 2\alpha=2\sin\alpha\cos\alpha$$

$$\frac{dy}{dx}=-2$$

(ii) $x=\sqrt{1+\sec t}$, **and** $y=\sqrt{\tan t}$ **12302023**

Solution:

$x=\sqrt{1+\sec t}$ .....(i)

$y=\sqrt{\tan t}$ .....(ii)

From equation (i)

$$x=\sqrt{1+\sec t}$$

Differentiate w.r.t. 't':

$$\frac{d}{dt}(x)=\frac{d}{dt}\left(\sqrt{1+\sec t}\right)$$

$$\frac{dx}{dt}=\frac{d}{dt}\left(1+\sec t\right)^{\frac{1}{2}}$$

$$\frac{dx}{dt}=\frac{1}{2}\left(1+\sec t\right)^{-1/2}\frac{d}{dt}\left(1+\sec t\right)$$

$$=\frac{1}{2\sqrt{1+\sec t}}\left(0+\sec t\tan t\right)$$

$$\frac{dx}{dt}=\frac{\sec t\tan t}{2\sqrt{1+\sec t}}$$

Now,

From equation (ii)

$$y=\sqrt{\tan t}$$

Differentiate w.r.t. 't':

$$\frac{d}{dt}(y)=\frac{d}{dt}\left(\sqrt{\tan t}\right)$$

$$\frac{dy}{dt}=\frac{d}{dt}\left(\tan t\right)^{1/2}$$

$$\frac{dy}{dt}=\frac{1}{2}\left(\tan t\right)^{-1/2}\frac{d}{dt}\left(\tan t\right)$$

$$\frac{dy}{dt}=\frac{\sec^2 t}{2\sqrt{\tan t}}$$

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By Chain Rule,

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$=\frac{\sec^2 t}{2\sqrt{\tan t}}\cdot\frac{2\sqrt{1+\sec t}}{\sec t\tan t}$$

$$=\frac{\sec t\sqrt{1+\sec t}}{\tan t\sqrt{\tan t}}$$

$$=\frac{\sec t\cdot\sqrt{1+\sec t}}{(\tan t)(\tan t)^{1/2}}$$

$$\frac{dy}{dx}=\frac{\sec t\sqrt{1+\sec t}}{(\tan t)^{3/2}}$$

14. If $x=a\sec t$ and $y=b\tan t$, show that

$$\frac{dy}{dx}=\frac{b^2x}{a^2y}.$$ **12302024**

Solution:

Given: $x=a\sec t$ .....(i)

Differentiate w.r.t. 't':

$$\frac{d}{dt}(x)=\frac{d}{dt}(a\sec t)$$

$$\frac{dx}{dt}=a\sec t\tan t$$

$$\Rightarrow\frac{dt}{dx}=\frac{1}{a\sec t\tan t}$$

Given: $y=b\tan t$ .....(ii)

Differentiate w.r.t. 't':

$$\frac{d}{dt}(y)=\frac{d}{dt}(b\tan t)$$

$$\frac{dy}{dt}=b\sec^2 t$$

By Chain Rule,

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$=b\sec^2 t\cdot\frac{1}{a\sec t\tan t}$$

$$=\frac{b\sec^2 t}{a\sec t\tan t}$$

$$\frac{dy}{dx}=\frac{b\sec t}{a\tan t}$$ .....(iii)

From equation (i), we have:

$$\boxed{\sec t=\frac{x}{a}}$$

From equation (ii), we have:

$$\boxed{\tan t=\frac{y}{b}}$$

Putting in equation (iii), we have:

$$\frac{dy}{dx}=\frac{b}{a}\left(\frac{\frac{x}{a}}{\frac{y}{b}}\right)=\frac{b}{a}\left[\frac{x}{a}\times\frac{b}{y}\right]$$

$$\frac{dy}{dx}=\frac{b^2x}{a^2y}$$

Hence proved.

15. Use implicit differentiation to find $\frac{dy}{dx}$ if:

(i) $ax^2+by^2+2gx+2fy+c=0$ **12302025**

Solution:

$$ax^2+by^2+2gx+2fy+c=0$$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(ax^2+by^2+2gx+2fy+c)=0$$

$$\frac{d}{dx}(ax^2)+\frac{d}{dx}(by^2)+\frac{d}{dx}(2gx)+\frac{d}{dx}(2fy)+\frac{d}{dx}(c)=0$$

$$a(2x)+b\left(2y\frac{dy}{dx}\right)+2g(1)+2f\left(\frac{dy}{dx}\right)+0=0$$

$$2ax+2by\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0$$

$$2by\frac{dy}{dx}+2f\frac{dy}{dx}=-2ax-2g$$

$$(2by+2f)\frac{dy}{dx}=-2ax-2g$$

$$\frac{dy}{dx}=\frac{-2(ax+g)}{2(by+f)}$$

$$\boxed{\frac{dy}{dx}=\frac{-(ax+g)}{(by+f)}}$$

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(ii) $x+\tan(xy)=0$ **12302026**

Solution:

$x+\tan(xy)=0$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}\left[x+\tan(xy)\right]=0$$

$$\frac{d}{dx}(x)+\frac{d}{dx}\tan(xy)=0$$

$$(1)+\sec^2(xy)\frac{d}{dx}(xy)=0$$

$$1+\sec^2(xy)\left[x\frac{dy}{dx}+y\cdot 1\right]=0$$

$$\sec^2(xy)\left(x\frac{dy}{dx}+y\right)=-1$$

$$\left(x\frac{dy}{dx}+y\right)=-\frac{1}{\sec^2(xy)}$$

$$x\frac{dy}{dx}+y=-\cos^2(xy)$$

$$x\frac{dy}{dx}=-y-\cos^2(xy)$$

$$\frac{dy}{dx}=\frac{-y-\cos^2(xy)}{x}$$

$$\boxed{\frac{dy}{dx}=-\left[\frac{y+\cos^2(xy)}{x}\right]}$$

16. Differentiate w.r.t. the variables involved.

(i) $\sin^{-1}\left(\frac{x}{a}\right)$ **12302027**

Solution:

$\sin^{-1}\left(\frac{x}{a}\right)$

Let $y=\sin^{-1}\left(\frac{x}{a}\right)$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y)=\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\frac{d}{dx}\left(\frac{x}{a}\right)$$

$$\boxed{\because \frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}}$$

$$=\frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}}\left(\frac{1}{a}\right)$$

$$=\frac{1}{\frac{\sqrt{a^2-x^2}}{a}}\left(\frac{1}{a}\right)$$

$$\frac{dy}{dx}=\frac{a}{\sqrt{a^2-x^2}}\left(\frac{1}{a}\right)$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{a^2-x^2}}$$

(ii) $\csc^{-1}\left(\frac{x^2+1}{x^2-1}\right)$ **12302028**

Solution:

$\csc^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

Let $y=\operatorname{cosec}^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y)=\frac{d}{dx}\operatorname{cosec}^{-1}\left(\frac{x^2+1}{x^2-1}\right)$$

$$\frac{dy}{dx}=\frac{-1}{\left(\frac{x^2+1}{x^2-1}\right)\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^2-1}}\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$\boxed{\because \frac{d}{dx}\operatorname{cosec}^{-1}x=\frac{-1}{x\sqrt{x^2-1}}}$$

$$\frac{dy}{dx}=\frac{-1}{\left(\frac{x^2+1}{x^2-1}\right)\sqrt{\frac{\left(x^2+1\right)^2-\left(x^2-1\right)^2}{\left(x^2-1\right)^2}}}\left[\frac{\left(x^2-1\right)(2x)-\left(x^2+1\right)(2x)}{\left(x^2-1\right)^2}\right]$$

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$$\frac{dy}{dx}=\frac{-1}{\frac{\left(x^2+1\right)}{\left(x^2-1\right)^2}\frac{\sqrt{x^4+1+2x^2-x^4-1+2x^2}}{\left(x^2-1\right)}}\left(\frac{-4x}{\left(x^2-1\right)^2}\right)$$

$$\frac{dy}{dx}=\frac{-1}{\frac{\left(x^2+1\right)\left(\sqrt{4x^2}\right)}{\left(x^2-1\right)^2}}\cdot\frac{-4x}{(x-1)^2}$$

$$\frac{dy}{dx}=\frac{\left(x^2-1\right)^2}{\left(x^2+1\right)(2x)}\cdot\frac{4x}{\left(x^2-1\right)^2}$$

$$\frac{dy}{dx}=\frac{2}{x^2+1}$$

(iii) $\cot^{-1}\left(\sqrt{x}\right)+\sqrt{\cot^{-1}x}$ **12302029**

Solution:

$\cot^{-1}\left(\sqrt{x}\right)+\sqrt{\cot^{-1}x}$

Let $y=\cot^{-1}\left(\sqrt{x}\right)+\sqrt{\cot^{-1}x}$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y)=\frac{d}{dx}\left[\cot^{-1}\sqrt{x}+\sqrt{\cot^{-1}x}\right]$$

$$\frac{dy}{dx}=\frac{d}{dx}\left(\cot^{-1}\sqrt{x}\right)+\frac{d}{dx}\left(\sqrt{\cot^{-1}x}\right)$$

$$\frac{dy}{dx}=\frac{-1}{1+\left(\sqrt{x}\right)^2}\frac{d}{dx}\sqrt{x}+\frac{1}{2}\left(\cot^{-1}x\right)^{-1/2}\frac{d}{dx}\left(\cot^{-1}x\right)$$

$$\boxed{\because \cot^{-1}x=\frac{-1}{1+x^2}}$$

$$\frac{dy}{dx}=\frac{-1}{1+x}\cdot\frac{1}{2}(x)^{-1/2}+\frac{1}{2\sqrt{\cot^{-1}x}}\left(\frac{-1}{1+x^2}\right)$$

$$\frac{dy}{dx}=\frac{-1}{2\sqrt{x}(1+x)}-\frac{1}{2\left(1+x^2\right)\sqrt{\cot^{-1}x}}$$

$$\boxed{\frac{dy}{dx}=\frac{-1}{2}\left[\frac{1}{\sqrt{x}(1+x)}+\frac{1}{\left(1+x^2\right)\sqrt{\cot^{-1}x}}\right]}$$

(iv) $\left(\tan t\right)^{-1}+\frac{1}{\tan^{-1}t}$ **12302030**

Solution:

$\left(\tan t\right)^{-1}+\frac{1}{\tan^{-1}t}$

Let $y=\left(\tan t\right)^{-1}+\frac{1}{\tan^{-1}t}$

Differentiate w.r.t. 't':

$$\frac{d}{dt}(y)=\frac{d}{dt}\left[\left(\tan t\right)^{-1}+\frac{1}{\tan^{-1}t}\right]$$

$$\frac{dy}{dt}=\frac{d}{dt}\left(\tan t\right)^{-1}+\frac{d}{dt}\left(\tan^{-1}t\right)^{-1}$$

$$\frac{dy}{dt}=(-1)\left(\tan t\right)^{-2}\frac{d}{dt}\left(\tan t\right)+\frac{d}{dt}\left(\tan^{-1}t\right)^{-1}$$

$$=(-1)\left(\tan t\right)^{-2}\left(\sec^2 t\right)+(-1)\left(\tan^{-1}t\right)^{-2}\frac{d}{dt}\left(\tan^{-1}t\right)$$

$$=\frac{-\sec^2 t}{\left(\tan t\right)^2}-\frac{1}{\left(\tan^{-1}t\right)^2}\left(\frac{1}{1+t^2}\right)$$

$$\boxed{\frac{dy}{dt}=-\left(\frac{\sec^2 t}{\left(\tan t\right)^2}+\frac{1}{\left(\tan^{-1}t\right)^2\left(1+t^2\right)}\right)}$$

Derivative of Exponential Functions

A function defined by

$f(x)=a^x$, where $a>0$ and $a\neq 1$

is called an exponential function.

If $a=e\approx 2.71828$ then $f(x)=e^x$, is called natural exponential function.

Now, we shall find the derivative of $f(x)=a^x$, using the definition.

Let $f(x)=a^x$ .....(i)

From equation (i) we have

$f(x+h)=a^{x+h}$ .....(ii)

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

(definition of derivative)

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After substituting the expression for $f(x)$ and $f(x+h)$ from equation (i) and (ii), we get:

$$= \lim_{h\to 0}\frac{a^{x+h}-a^x}{h}$$

$$= \lim_{h\to 0}\frac{a^x\left(a^h-1\right)}{h}$$

$$= a^x \lim_{h\to 0}\frac{a^h-1}{h} \qquad \boxed{\because \lim_{h\to 0}\frac{a^h-1}{h}=\ln a}$$

$$= a^x \ln a$$

Hence, we have proved that

$$\boxed{\frac{d}{dx}\left(a^x\right)=a^x\ln a}$$

Example 7: Find $\frac{dy}{dx}$ if:

(i) $y=e^{\sin 2x}$ **12302031**

(ii) a^{x^3} **12302032**

Solution:

(i) $y=e^{\sin 2x}$

Differentiate w.r.t. x:

$$\frac{d}{dx}(y)=\frac{d}{dx}\left(e^{\sin 2x}\right)$$

$$\frac{dy}{dx}=e^{\sin^2 x}\frac{d}{dx}\left(\sin^2 x\right) \qquad \boxed{\because \frac{d}{dx}\left(e^u\right)=e^u\cdot\frac{du}{dx}}$$

$$\frac{dy}{dx}=e^{\sin^2 x}2\sin x\frac{d}{dx}(\sin x)=2\sin x e^{\sin^2 x}\cos x$$

$$\frac{dy}{dx}=2\sin x\cos x e^{\sin^2 x}$$

(ii) $y=a^{x^3}$

Differentiate $y=a^{x^3}$ w.r.t. x:

$$\frac{d}{dx}(y)=\frac{d}{dx}\left(a^{x^3}\right)$$

$$\frac{dy}{dx}=a^{x^3}\ln a\cdot\frac{d}{dx}\left(x^3\right) \qquad \boxed{\because \left(a^u\right)'=a^u\ln a\cdot\frac{du}{dx}}$$

$$\frac{dy}{dx}=a^{x^3}\ln a\left(3x^2\right)$$

$$\frac{dy}{dx}=3x^2a^{x^3}\ln a$$

Derivative of Logarithmic Functions

If $a^y=x$, where $a>0$ then $a\neq 1$, then y is called the logarithm of x to the base a, written as:

$$y=\log_a x \ , \quad x>0 \ , \quad y\in\mathbb{R}$$

If $a=e$, where $e\approx 2.71828$, then the function

$y=\log_e x=\ln x$ (whenever $e^y=x$)

is called the natural logarithm of x.

Now, we will find the derivative of $f(x)=\log_a x$, using the definition.

Let $f(x)=\log_a x$ .....(i)

From equation (i) we have:

$f(x+h)=\log_a(x+h)$ .....(ii)

We know

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

(definition of derivative)

After substituting the expression for $f(x)$ and $f(x+h)$ from equation (i) and (ii), we get:

$$= \lim_{h\to 0}\frac{\log_a(x+h)-\log_a x}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\log_a\left(\frac{x+h}{x}\right)$$

$$= \lim_{h\to 0}\frac{1}{x}\cdot\frac{x}{h}\log_a\left(1+\frac{h}{x}\right)$$

$$= \lim_{h\to 0}\frac{1}{x}\log_a\left(1+\frac{h}{x}\right)^{\frac{x}{h}} \qquad \boxed{\because \lim_{h\to 0}\left(1+\frac{h}{a}\right)^{\frac{a}{h}}=e}$$

$$= \frac{1}{x}\cdot\log_a e$$

$$= \frac{1}{x}\cdot\frac{1}{\log_e a} \qquad \boxed{\because \log_a b=\frac{1}{\log_b a}}$$

$$= \frac{1}{x\ln a} \qquad \boxed{\because \log_e x=\ln x}$$

Hence, we have proved that

$$\frac{d}{dx}\left(\log_a x\right)=\frac{1}{x\ln a}.$$

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Example 8: **Find** $\frac{dy}{dx}$ **if** $y=\frac{a^x}{\log_a x}$ 12302033

Solution: Differentiate $y=\frac{a^x}{\log_a x}$ with respect to 'x':

$$\frac{d}{dx}(y)=\frac{d}{dx}\left[\frac{a^x}{\log_a x}\right]$$

$$\frac{dy}{dx}=\frac{\left(a^x\right)'\log_a x-a^x\left(\log_a x\right)'}{\left(\log_a x\right)^2}$$

$$\boxed{\because\left(\frac{u}{v}\right)'=\frac{vu'-uv'}{(v)^2}}$$

$$=\frac{a^x\ln a\log_a x-a^x\cdot\frac{1}{x\ln a}}{\left(\log_a x\right)^2}$$

$$\boxed{=\frac{a^x\left[x(\ln a)^2\log_a x-1\right]}{x\ln a\left(\log_a x\right)^2}}$$

Example 9: **Find** $\frac{dy}{dx}$ **if** $y=\ln\left(\cos\left(e^x\right)\right)$. 12302034

Solution: Differentiate $y=\ln\left(\cos\left(e^x\right)\right)$. with respect to '$x$':

$$\frac{d}{dx}(y)=\frac{d}{dx}\left[\ln\cos\left(e^x\right)\right]$$

$$\frac{dy}{dx}=\frac{1}{\cos\left(e^x\right)}\frac{d}{dx}\left(\cos e^x\right)$$

$$=\frac{1}{\cos\left(e^x\right)}\left(-\sin e^x\right)\frac{d}{dx}\left(e^x\right)$$

$$=-\frac{\sin\left(e^x\right)}{\cos\left(e^x\right)}e^x$$

$$\boxed{=-e^x\tan\left(e^x\right)}$$

EXERCISE 2.2

1. Find $f'(x)$ **if:**

(i) $f(x)=\left(x^3+4x\right)e^{1/x}$ 12302035

Solution:

$$f(x)=\left(x^3+4x\right)e^{1/x}$$

Differentiate w.r.t. 'x':

$$f'(x)=\frac{d}{dx}\left[\left(x^3+4x\right)e^{1/x}\right]$$

$$f'(x)=\left(x^3+4x\right)\frac{d}{dx}\left(e^{1/x}\right)+\left(e^{1/x}\right)\frac{d}{dx}\left(x^3+4x\right)$$

$$\boxed{\because(uv)'=uv'+vu'}$$

$$=\left(x^3+4x\right)\left(e^{1/x}\left(-\frac{1}{x^2}\right)\right)+\left(e^{1/x}\right)\left(3x^2+4\right)$$

$$=e^{1/x}\left(\frac{-\left(x^3+4x\right)}{x^2}+3x^2+4\right)$$

$$f'(x)=e^{1/x}\left(\frac{-x^3-4x+3x^4+4x^2}{x^2}\right)$$

$$f'(x)=e^{1/x}\left(\frac{3x^4-x^3+4x^2-4x}{x^2}\right)$$

(ii) $f(x)=\sin\left(2^x\right)$ 12302036

Solution:

$$f(x)=\sin\left(2^x\right)$$

Differentiate w.r.t. 'x':

$$f'(x)=\frac{d}{dx}\left[\sin\left(2^x\right)\right]$$

$$f'(x)=\cos\left(2^x\right)\frac{d}{dx}\left(2^x\right)$$

$$\boxed{\because\frac{d}{dx}\left(a^x\right)=a^x\ln a}$$

$$f'(x)=\cos\left(2^x\right)\left(2^x\ln 2\right)$$

$$f'(x)=2^x\ln 2\cos\left(2^x\right)$$

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(iii) $f(x)=e^{x\cos x}$ **12302037**

Solution:

$f(x)=e^{x\cos x}$

Differentiate w.r.t. 'x':

$$f'(x)=\frac{d}{dx}\left(e^{x\cos x}\right)$$

$$f'(x)=e^{x\cos x}\frac{d}{dx}(x\cos x)$$

$$f'(x)=e^{x\cos x}\left[x\frac{d}{dx}(\cos x)+\cos x\frac{d}{dx}(x)\right]$$

$$=e^{x\cos x}\left[x(-\sin x)+\cos x\cdot 1\right]$$

$$f'(x)=e^{x\cos x}(-x\sin x+\cos x)$$

$$\boxed{f'(x)=e^{x\cos x}(\cos x-x\sin x)}$$

(iv) $f(x)=\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$ **12302038**

Solution:

$$f(x)=\frac{e^x+e^{-x}}{e^x-e^{-x}}$$

Differentiate w.r.t. to 'x':

$$f'(x)=\frac{d}{dx}\left[\frac{e^x+e^{-x}}{e^x-e^{-x}}\right]$$

$$f'(x)=\frac{\left(e^x-e^{-x}\right)\frac{d}{dx}\left(e^x+e^{-x}\right)-\left(e^x+e^{-x}\right)\frac{d}{dx}\left(e^x-e^{-x}\right)}{\left(e^x-e^{-x}\right)^2}$$

$$\boxed{\because\left(\frac{u}{v}\right)'=\frac{vu'-uv'}{(v)^2}}$$

$$f'(x)=\frac{\left(e^x-e^{-x}\right)\left[e^x+e^{-x}(-1)\right]-\left(e^x+e^{-x}\right)\left[e^x-e^{-x}(-1)\right]}{\left(e^x-e^{-x}\right)^2}$$

$$f'(x)=\frac{\left(e^x-e^{-x}\right)\left(e^x-e^{-x}\right)-\left(e^x+e^{-x}\right)\left(e^x+e^{-x}\right)}{\left(e^x-e^{-x}\right)^2}$$

$$=\frac{\left(e^x-e^{-x}\right)^2-\left(e^x+e^{-x}\right)^2}{\left(e^x-e^{-x}\right)^2}$$

$$=\frac{\left(e^{2x}+e^{-2x}-2e^xe^{-x}\right)-\left(e^{2x}+e^{-2x}+2e^xe^{-x}\right)}{\left(e^x-e^{-x}\right)^2}$$

$$=\frac{e^{2x}+e^{-2x}-2-e^{2x}-e^{-2x}-2}{\left(e^x-e^{-x}\right)^2}$$

$$f'(x)=\frac{-4}{\left(e^x-e^{-x}\right)^2}$$

2. Find $\dfrac{dy}{dx}$ if:

(i) $y=\ln\left(e^x-e^{-x}\right)$ **12302039**

Solution:

$$y=\ln\left(e^x-e^{-x}\right)$$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y)=\frac{d}{dx}\left[\ln\left(e^x-e^{-x}\right)\right]$$

$$=\frac{1}{e^x-e^{-x}}\frac{d}{dx}\left(e^x-e^{-x}\right)$$

$$=\frac{1}{e^x-e^{-x}}\cdot\left[e^x-e^{-x}(-1)\right]$$

$$\frac{dy}{dx}=\frac{e^x+e^{-x}}{e^x-e^{-x}}$$

(ii) $y=\log_{10}\sqrt{\dfrac{1-x^2}{1+x^2}}$ **12302040**

Solution:

$$y=\log_{10}\sqrt{\frac{1-x^2}{1+x^2}}$$

$$y=\log_{10}\left(\frac{1-x^2}{1+x^2}\right)^{1/2}$$

$$y=\frac{1}{2}\log_{10}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\boxed{\because\log_a\left(\frac{m}{n}\right)=\log_a m-\log_a n}$$

$$y=\frac{1}{2}\left[\log_{10}\left(1-x^2\right)-\log_{10}\left(1+x^2\right)\right]$$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y)=\frac{1}{2}\frac{d}{dx}\left[\log_{10}\left(1-x^2\right)-\log_{10}\left(1+x^2\right)\right]$$

$$\frac{dy}{dx}=\frac{1}{2}\left[\frac{d}{dx}\log_{10}\left(1-x^2\right)-\frac{d}{dx}\log_{10}\left(1+x^2\right)\right]$$

$$\frac{dy}{dx}=\frac{1}{2}\left[\left(\frac{1}{\left(1-x^2\right)\ln 10}\frac{d}{dx}\left(1-x^2\right)\right)-\left(\frac{1}{\left(1+x^2\right)\ln 10}\frac{d}{dx}\left(1+x^2\right)\right)\right]$$

$$=\frac{1}{2}\left[\frac{-2x}{\left(1-x^2\right)\ln 10}-\frac{2x}{\left(1+x^2\right)\ln 10}\right]$$

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$$= \frac{2}{2}\left[\frac{-x}{\left(1-x^2\right)\ln 10} - \frac{x}{\left(1+x^2\right)\ln 10}\right]$$

$$= \frac{-x}{\left(1-x^2\right)\ln 10} - \frac{x}{\left(1+x^2\right)\ln 10}$$

$$= \frac{-x\left(1+x^2\right) - x\left(1-x^2\right)}{\left(1-x^2\right)\left(1+x^2\right)\ln 10}$$

$$= \frac{-x - x^3 - x + x^3}{\left[(1)^2 - \left(x^2\right)^2\right]\ln 10}$$

$$\frac{dy}{dx} = \frac{-2x}{\left(1-x^4\right)\ln 10}$$

(iii) $y = \ln\left(x - \sqrt{x^2-1}\right)$ **12302041**

Solution:

$$y = \ln\left(x - \sqrt{x^2-1}\right)$$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y) = \frac{d}{dx}\left[\ln\left(x - \sqrt{x^2-1}\right)\right]$$

$$\frac{dy}{dx} = \frac{1}{x - \sqrt{x^2-1}}\frac{d}{dx}\left(x - \sqrt{x^2-1}\right)$$

$$\boxed{\because \frac{d}{dx}(\ln x) = \frac{1}{x}}$$

$$= \frac{1}{x - \sqrt{x^2-1}}\left[\frac{d}{dx}(x) - \frac{d}{dx}\left(\sqrt{x^2-1}\right)\right]$$

$$= \frac{1}{x - \sqrt{x^2-1}}\left[1 - \frac{1}{2}\left(x^2-1\right)^{-1/2}(2x)\right]$$

$$= \frac{1}{x - \sqrt{x^2-1}}\left(1 - \frac{x}{\sqrt{x^2-1}}\right)$$

$$= \frac{1}{x - \sqrt{x^2-1}}\left(\frac{\sqrt{x^2-1} - x}{\sqrt{x^2-1}}\right)$$

$$= \frac{-\left(x - \sqrt{x^2-1}\right)}{\left(x - \sqrt{x^2-1}\right)\left(\sqrt{x^2-1}\right)}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{x^2-1}}$$

(iv) $y = \ln\left(1 - \cos^2 x\right)$ **12302042**

Solution:

$$y = \ln\left(1 - \cos^2 x\right)$$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y) = \frac{d}{dx}\left[\ln\left(1 - \cos^2 x\right)\right]$$

$$\frac{dy}{dx} = \frac{1}{1-\cos^2 x}\frac{d}{dx}\left(1 - \cos^2 x\right)$$

$$= \frac{1}{1-\cos^2 x}\left[\frac{d}{dx}(1) - \frac{d}{dx}\left(\cos^2 x\right)\right]$$

$$= \frac{1}{1-\cos^2 x}\cdot\left[0 - 2\cos x(-\sin x)\right]$$

$$= \frac{2\sin x\cos x}{1-\cos^2 x}$$

$$= \frac{2\sin x\cos x}{\sin^2 x} \quad \boxed{\because 1 - \cos^2 x = \sin^2 x}$$

$$= \frac{2\cos x}{\sin x}$$

$$\frac{dy}{dx} = 2\cot x$$

(v) $y = \frac{\ln x}{x^2}$ **12302043**

Solution:

$$y = \frac{\ln x}{x^2}$$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{\ln x}{x^2}\right)$$

$$\frac{dy}{dx} = \frac{x^2\frac{d}{dx}(\ln x) - \ln x\frac{d}{dx}\left(x^2\right)}{\left(x^2\right)^2}$$

$$\boxed{\because \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}}$$

$$\frac{dy}{dx} = \frac{x^2\left(\frac{1}{x}\right) - \ln x(2x)}{(x)^4}$$

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$$\frac{dy}{dx}=\frac{x-2x\ln x}{(x)^4}$$

$$\frac{dy}{dx}=\frac{x(1-2\ln x)}{x^4}$$

$$\frac{dy}{dx}=\frac{1-2\ln x}{x^3}$$

(vi) $y=\dfrac{\sqrt{x^2-1}(2x+1)}{(x^3-1)^{3/2}}$ **12302044**

Solution:

$$y=\frac{\sqrt{x^2-1}(2x+1)}{(x^3-1)^{3/2}}$$

$$y=\frac{\sqrt{(x+1)(x-1)}(2x+1)}{\left[(x-1)(x^2+x+1)\right]^{3/2}}$$

$$\because (a^2-b^2)=(a+b)(a-b)$$
$$\because (a^3-b^3)=(a-b)(a^2+ab+b^2)$$

$$y=\frac{\sqrt{x+1}\cdot\sqrt{x-1}(2x+1)}{(x-1)^{3/2}(x^2+x+1)^{3/2}}$$

$$y=\frac{(x+1)^{1/2}\cdot(x-1)^{1/2}\cdot(2x+1)}{(x-1)^{1/2}\cdot(x-1)(x^2+x+1)^{3/2}}$$

$$y=\frac{(x+1)^{1/2}(2x+1)}{(x-1)(x^2+x+1)^{3/2}}$$

Taking 'ln' on both sides:

$$\ln y=\ln\left[\frac{(x+1)^{1/2}(2x+1)}{(x-1)(x^2+x+1)^{3/2}}\right]$$

$$\ln y=\ln(x+1)^{1/2}+\ln(2x+1)-\ln(x-1)-\ln(x^2+x+1)^{3/2}$$

$$\ln y=\frac{1}{2}\ln(x+1)+\ln(2x+1)-\ln(x-1)-\frac{3}{2}\ln(x^2+x+1)$$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(\ln y)=\frac{d}{dx}\left[\frac{1}{2}\ln(x+1)+\ln(2x+1)-\ln(x-1)-\frac{3}{2}\ln(x^2+x+1)\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}\ln(x+1)+\frac{d}{dx}\ln(2x+1)-\frac{d}{dx}\ln(x-1)-\frac{3}{2}\frac{d}{dx}\ln(x^2+x+1)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left(\frac{1}{x+1}\right)+\frac{1}{2x+1}(2)-\frac{1}{x-1}-\frac{3}{2}\left[\frac{1}{x^2+x+1}(2x+1)\right]$$

$$\frac{dy}{dx}=y\left[\frac{1}{2(x+1)}+\frac{2}{2x+1}-\frac{1}{x-1}-\frac{(6x+3)}{2(x^2+x+1)}\right]$$

$$\frac{dy}{dx}=\frac{(x+1)^{1/2}(2x+1)}{(x-1)(x^2+x+1)^{3/2}}\left[\frac{1}{2(x+1)}+\frac{2}{2x+1}-\frac{1}{x-1}-\frac{(6x+3)}{2(x^2+x+1)}\right]$$

Equations of Tangent and Normal Lines to a Curve

Tangent Line: Suppose that x_0 is a point in the domain of the function $f(x)$. The tangent line to the curve $y=f(x)$ at the point $(x_0,f(x_0))$ is the line with equation.

$$y-y_0=m(x-x_0)$$

Where $m=f'(x_0)=\left.\dfrac{dy}{dx}\right|_{x=x_0}$,

provided $f'(x_0)$ exists.

Need to Know!

Vertical Tangent

If m is ∞ or $-\infty$ at $x=x_0$ then $x=x_0$ is tangent to the curve $y=f(x)$ is called vertical tangent.

Normal Line: A line through point $(x_0,f(x_0))$ of the curve $y=f(x)$ and perpendicular to tangent line is called normal line to the curve $y=f(x)$.

If $m=f'(x_0)$ is non zero, then equation of the normal line is $y-y_0=-\dfrac{1}{m}(x-x_0)$ and if $m=f'(x_0)=0$ then $x=x_0$ is the equation of the normal line.

Note: Slope of tangent line to a curve at a point is also known as slope of the curve at the point.

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Example 10: **Find the equations of the tangent and normal to the curve** $y = \sec x$ **at the point where** $x = \frac{\pi}{4}$. **12302045**

Solution: Here, $y = \sec x$ .....(i)

Differentiate (i) with respect to x:

$$\frac{d}{dx}(y) = \frac{d}{dx}(\sec x)$$

$$\frac{dy}{dx} = \sec x \tan x$$

$$m = \left.\frac{dy}{dx}\right|_{x=\pi/4} = \sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)$$

$$= \left(\sqrt{2}\right)(1) = \sqrt{2} \text{ (Slope of tangent line)}$$

Put $x = \frac{\pi}{4}$ in (i),

$$y = \sec\left(\frac{\pi}{4}\right) = \sqrt{2}$$

The equation of tangent at $\left(\frac{\pi}{4}, \sqrt{2}\right) = (x_0, y_0)$ with slope $m = \sqrt{2}$ is:

$$y - \sqrt{2} = \sqrt{2}\left(x - \frac{\pi}{4}\right) \quad \boxed{\because y - y_0 = m(x - x_0)}$$

$$y - \sqrt{2} = \sqrt{2}x - \frac{\pi\sqrt{2}}{4}$$

Multiply both sides by '4'

$$4y - 4\sqrt{2} = 4\sqrt{2}x - \pi\sqrt{2}$$

$$4\sqrt{2}x - 4y + 4\sqrt{2} - \pi\sqrt{2} = 0$$

The equation of normal to given curve at $\left(\frac{\pi}{4}, \sqrt{2}\right) = (x_0, y_0)$ is:

$$y - \sqrt{2} = -\frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right)$$

$$\boxed{\because y - y_0 = \frac{-1}{m}(x - x_0)}$$

$$y - \sqrt{2} = \frac{-x}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}$$

Multiply both sides by $4\sqrt{2}$

$$4\sqrt{2}y - 8 = -4x + \pi$$

$$4x + 4\sqrt{2}y - 8 - \pi = 0$$

Higher-Order Derivatives

The derivative f' of a function f is itself a function and hence may have a derivative of its own. If f' is differentiable, then its derivative is denoted by f'' and is called the second derivative of f. As long as the new function remains differentiable, we can continue the process of differentiating to obtain the third, fourth, fifth and even higher derivatives of f. These successive derivatives are called higher order derivatives.

The following are the commonly used notations for higher-order derivatives:

1st derivative	2nd derivative	3rd derivative	...	n^{th} derivative
y'	y''	y'''	...	$y^{(n)}$
$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	$\frac{d^3y}{dx^3}$	...	$\frac{d^ny}{dx^n}$
$f'(x)$	$f''(x)$	$f'''(x)$	...	$f^{(n)}(x)$
y_1	y_2	y_3	...	y_n
Dy	D^2y	D^3y	...	D^ny
$\frac{df}{dx}$	$\frac{d^2f}{dx^2}$	$\frac{d^3f}{dx^3}$	...	$\frac{d^nf}{dx^n}$

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Example 11: Find the higher-order derivative of the function. 12302046

$$f(x)=\frac{1}{4}x^4-x^3+\frac{1}{2}x^2-3$$

Solution:

Differentiate $f(x)=\frac{1}{4}x^4-x^3+\frac{1}{2}x^2-3$ to find its higher-order derivatives.

$$f'(x)=\frac{1}{4}\cdot 4x^3-3x^2+\frac{1}{2}\cdot 2x-0$$
$$f'(x)=x^3-3x^2+x$$
$$f''(x)=3x^2-6x+1$$
$$f'''(x)=6x-6$$
$$f^{(4)}(x)=6$$
$$f^{(5)}(x)=0$$
$$\vdots$$
$$f^{(n)}(x)=0\,(n\geq 5)$$

Example 12: If $x=e^{at}$ and $y=e^{2at}$, show that $x^2\frac{d^2y}{dx^2}-2y=0$. 12302047

Solution:

Given: $x=e^{at}$ .....(i)

and $y=e^{2at}$ .....(ii)

Differentiate equations (i) and (ii) with respect to 't', to get:

$$\frac{d}{dt}(x)=\frac{d}{dt}\left(ae^{at}\right) \text{ and } \frac{d}{dt}(y)=\frac{d}{dt}\left(2ae^{2at}\right)$$
$$\frac{dx}{dt}=ae^{at} \quad \text{and} \quad \frac{dy}{dt}=2ae^{2at}$$

Using equations (i) and (ii), we have:

$$\frac{dx}{dt}=ax \quad \text{and} \quad \frac{dy}{dt}=2ay$$

Apply the Chain Rule to obtain:

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$
$$\frac{dy}{dx}=\frac{2ay}{ax}$$
$$\frac{dy}{dx}=\frac{2y}{x} \quad \text{.....(iii)}$$

Again, Differentiate with respect to 'x':

$$\frac{d^2y}{dx^2}=2\frac{d}{dx}\left(\frac{y}{x}\right)$$
$$\frac{d^2y}{dx^2}=2\left(\frac{\frac{dy}{dx}x-y\cdot 1}{x^2}\right) \quad \boxed{\because \left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}}$$

Using equation (iii), we get:

$$\frac{d^2y}{dx^2}=2\left(\frac{\frac{2y}{x}x-y}{x^2}\right)$$
$$\frac{d^2y}{dx^2}=2\left(\frac{y}{x^2}\right)$$
$$x^2\frac{d^2y}{dx^2}=2y$$
$$\Rightarrow x^2\frac{d^2y}{dx^2}-2y=0$$

Hence proved.

Example 13: Find all higher-order derivative of the function $y=e^{ax}$. 12302048

Solution: We differentiate $y=e^{ax}$ to find its higher-order derivatives.

$$\frac{dy}{dx}=e^{ax}\cdot\frac{d}{dx}(ax)$$
$$\frac{dy}{dx}=e^{ax}\cdot a=ae^{ax}$$

Differentiate with respect to 'x', to obtain:

$$\frac{d^2y}{dx^2}=a\frac{d}{dx}\left(e^{ax}\right)$$
$$\frac{d^2y}{dx^2}=a\left(e^{ax}a\right)=a^2e^{ax}$$
$$\frac{d^3y}{dx^3}=a^2\left(e^{ax}a\right)=a^3e^{ax}$$

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$$\frac{d^4y}{dx^4} = a^4e^{ax}$$
$$\vdots$$
$$\frac{d^ny}{dx^n} = a^ne^{ax} \quad (n \geq 4)$$
$$\frac{d^ny}{dx^n} = a^ny \quad \boxed{\because y = e^{ax}}$$

Example 14: Find y_4 if $y = \ln\sqrt{\frac{2x-3}{x+1}}$. 12302049

Solution: We are given: $y = \ln\sqrt{\frac{2x-3}{x+1}}$

Before differentiating we apply properties of logarithm:

$$y = \frac{1}{2}\ln\left(\frac{2x-3}{x+1}\right)$$

$$\boxed{\because \ln\left(\frac{m}{n}\right) = \ln m - \ln n}$$

$$= \frac{1}{2}\left[\ln(2x-3) - \ln(x+1)\right]$$

Differentiate w.r.t. 'x'

$$y_1 = \frac{1}{2}\left(\frac{2}{2x-3} - \frac{1}{x+1}\right)$$

$$y_1 = \frac{1}{2}\left((2x-3)^{-1} - (x+1)^{-1}\right)$$

Differentiate w.r.t. 'x'

$$y_2 = \frac{1}{2}\left[-2(2x-3)^{-2} + (x+1)^{-2}\right]$$

Differentiate w.r.t. 'x'

$$y_3 = \frac{1}{2}\left[4(2x-3)^{-3} - 2(x+1)^{-3}\right]$$

Differentiate w.r.t. 'x'

$$y_4 = \frac{1}{2}\left[-12(2x-3)^{-4} + 6(x+1)^{-4}\right]$$

$$y_4 = \frac{1}{2}\left[\frac{-12}{(2x-3)^4} + \frac{6}{(x+1)^4}\right]$$

$$= \frac{-6}{(2x-3)^4} + \frac{3}{(x+1)^4}$$

EXERCISE 2.3

1. Find the equation of the tangent line to the curve $y = \sin^{-1}\left(\frac{x}{2}\right) + \cos x$ at the point where $x = \frac{\pi}{2}$. 12302050

Solution:

Given: $y = \sin^{-1}\left(\frac{x}{2}\right) + \cos x$ .....(i)

Put $x = \frac{\pi}{2}$, in equation (i)

$$y = \sin^{-1}\left(\frac{\frac{\pi}{2}}{2}\right) + \cos\left(\frac{\pi}{2}\right)$$

$$y = \sin^{-1}\left(\frac{\pi}{4}\right) + 0 \qquad \boxed{\because \cos\frac{\pi}{2} = 0}$$

$$\boxed{y = \frac{1}{\sqrt{2}}}$$

So the point is $\left(\frac{\pi}{2}, \frac{1}{\sqrt{2}}\right)$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y) = \frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{2}\right) + \cos x\right]$$

$$\frac{dy}{dx} = \frac{d}{dx}\sin^{-1}\left(\frac{x}{2}\right) + \frac{d}{dx}(\cos x)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\frac{x^2}{4}}}\left(\frac{1}{2}\right) - \sin x$$

$$\boxed{\because \frac{d}{dx}\sin^{-1}A = \frac{1}{\sqrt{1-A^2}}}$$

$$= \frac{1}{\frac{\sqrt{4-x^2}}{2}}\left(\frac{1}{2}\right) - \sin x$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}} - \sin x$$

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$$m=\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}}=\frac{1}{\sqrt{4-\left(\frac{\pi}{2}\right)^2}}-\sin\left(\frac{\pi}{2}\right)$$

$$m=\frac{1}{\frac{\sqrt{16-\pi^2}}{2}}-1$$

$$m=\frac{2}{\sqrt{16-\pi^2}}-1$$

Now equation of tangent at $\left(\frac{\pi}{2},\frac{1}{\sqrt{2}}\right)$;

$$y-y_0=m(x-x_0)$$

$$y-\frac{1}{\sqrt{2}}=\left(\frac{2}{\sqrt{16-\pi^2}}-1\right)\left(x-\frac{\pi}{2}\right)$$

$$\boxed{\frac{\sqrt{2}y-1}{\sqrt{2}}=\left(\frac{2-\sqrt{16-x^2}}{\sqrt{16-\pi^2}}\right)\left(\frac{2x-\pi}{2}\right)}$$

2. Find a point on the curve $y=\tan^{-1}\left(\frac{x}{3}\right)-\frac{x}{3}$, where tangent is horizontal. Also find the equations of the tangent and normal at that point.

Solution: 12302051

Given:

$$y=\tan^{-1}\left(\frac{x}{3}\right)-\frac{x}{3} \quad \ldots\ldots\ldots(i)$$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y)=\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{3}\right)-\frac{x}{3}\right]$$

$$\frac{dy}{dx}=\frac{d}{dx}\tan^{-1}\left(\frac{x}{3}\right)-\frac{d}{dx}\left(\frac{x}{3}\right)$$

$$\frac{dy}{dx}=\frac{1}{1+\left(\frac{x}{3}\right)^2}\left(\frac{1}{3}\right)-\left(\frac{1}{3}\right)$$

$$\boxed{\because \frac{d}{dx}\tan^{-1}x=\frac{1}{1+x^2}}$$

$$=\frac{1}{1+\frac{x^2}{9}}\left(\frac{1}{3}\right)-\left(\frac{1}{3}\right)$$

$$\frac{dy}{dx}=\frac{9}{9+x^2}\left(\frac{1}{3}\right)-\left(\frac{1}{3}\right)$$

$$\frac{dy}{dx}=\frac{3}{9+x^2}-\left(\frac{1}{3}\right)$$

For horizontal tangent:

Put $\frac{dy}{dx}=0$

$$\frac{3}{x^2+9}-\frac{1}{3}=0$$

$$\frac{3}{x^2+9}=\frac{1}{3}$$

$$9=x^2+9$$

$$x^2=0$$

$x=0$ Put in equation (i) we get,

$$y=\tan^{-1}\left(\frac{0}{3}\right)-\left(\frac{0}{3}\right)$$

$$y=0$$

So, the point on given curve where tangent is horizontal is $(0,0)$.

Now equation of tangent at $(0,0)$:

$$y-y_0=m(x-x_0)$$

$$y-0=0(x-0)$$

$$\Rightarrow y=0$$

Normal is perpendicular to tangent. Slope of normal is undefined for slope zero.

So, its equation of normal line is: $x=0$.

3. Find a point on the curve $x+y-e^y=0$ where the tangent line is vertical. Also, find the equations of the tangent and normal at that point. 12302052

Solution:

Given:

$$x+y-e^y=0 \quad \ldots\ldots\ldots(i)$$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}\left(x+y-e^y\right)=0$$

$$\frac{d}{dx}(x)+\frac{d}{dx}(y)-\frac{d}{dx}\left(e^y\right)=0$$

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$$1+\frac{dy}{dx}-e^y\frac{dy}{dx}=0$$
$$\frac{dy}{dx}\left(1-e^y\right)=-1$$
$$\frac{dy}{dx}=\frac{-1}{1-e^y}$$

For vertical tangent: put $\frac{dy}{dx}=\infty$

i.e. denominator $=0$

$$\frac{-1}{1-e^y}=\frac{1}{0}$$
$$1-e^y=0$$
$$e^y=1$$
$$e^y=e^0$$

$y=0$ put in equation (i)

we get,

$$x+(0)-e^0=0$$
$$x-1=0$$
$$x=1$$

So, the point where tangent is vertical to curve is $(1,0)$

Equation of tangent:

Now equation of tangent at $(1,0)$ is:

$$y-y_0=\left.\frac{dy}{dx}\right|_p(x-x_0)$$
$$y-0=\left(\frac{-1}{1-e^0}\right)(x-1)$$
$$y=\frac{-1}{(0)}(x-1)$$
$$0=x-1$$

$x=1$ is the required equation of tangent.

Equation of Normal:

Now equation of normal at $(1,0)$ is:

$$y-y_0=\frac{-1}{\left(\frac{dy}{dx}\right)_p}(x-x_0)$$
$$y-0=\frac{-1}{\left(\frac{-1}{1-e^{(0)}}\right)}(x-1)$$
$$y-0=\left(1-e^0\right)(x-1)$$
$$y-0=(0)(x-1)$$

$y=0$ is the required equation of normal.

4. Find all points on the ellipse $x^2-xy+y^2=1$ at which the tangent line is horizontal. 12302053

Solution:

Given:

$$x^2-xy+y^2=1\ldots\ldots\ldots(i)$$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}\left(x^2-xy+y^2\right)=\frac{d}{dx}(1)$$
$$\frac{d}{dx}\left(x^2\right)-\frac{d}{dx}(xy)+\frac{d}{dx}\left(y^2\right)=0$$
$$2x-\left(x\frac{dy}{dx}+y\cdot 1\right)+2y\frac{dy}{dx}=0$$

$$\boxed{\because (uv)'=uv'+vu'}$$

$$2x-x\frac{dy}{dx}-y+2y\frac{dy}{dx}=0$$
$$(2y-x)\frac{dy}{dx}=y-2x$$
$$\boxed{\frac{dy}{dx}=\frac{y-2x}{2y-x}}$$

For horizontal tangent:

Put $\frac{dy}{dx}=0$

$$\frac{y-2x}{2y-x}=0$$
$$y-2x=0$$
$$y=2x\ \ldots\ldots\ldots(ii)$$

Put $y=2x$ in equation (i)

$$x^2-x(2x)+(2x)^2=1$$
$$x^2-2x^2+4x^2=1$$
$$3x^2=1$$
$$x^2=\frac{1}{3}$$
$$x=\pm\frac{1}{\sqrt{3}}$$

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Putting $x=\pm\frac{1}{\sqrt{3}}$ in equation (ii), we have:

$$y=2\left(\pm\frac{1}{\sqrt{3}}\right)$$

$$y=\pm\frac{2}{\sqrt{2}}$$

So, we have $\left(\pm\frac{1}{\sqrt{3}},\pm\frac{2}{\sqrt{3}}\right)$ which are required points.

5. Find y_2 if:

(i) $y=x^4-3x^2+4x-5$ **12302054**

Solution:

$y=x^4-3x^2+4x-5$

Differentiate w.r.t. to 'x':

$$y_1=\frac{d}{dx}\left(x^4-3x^2+4x-5\right)$$

$$y_1=\frac{d}{dx}\left(x^4\right)-3\frac{d}{dx}\left(x^2\right)+4\frac{d}{dx}(x)-\frac{d}{dx}(5)$$

$$y_1=4x^3-3(2x)+4(1)-0$$

$$y_1=4x^3-6x+4$$

Again Differentiate w.r.t. 'x':

$$y_2=\frac{d}{dx}\left(4x^3-6x+4\right)$$

$$y_2=4\frac{d}{dx}\left(x^3\right)-6\frac{d}{dx}(x)+\frac{d}{dx}(4)$$

$$y_2=4\left(3x^2\right)-6(1)+0$$

$$\boxed{y_2=12x^2-6}$$

(ii) $y=\ln(2x+3)$ **12302055**

Solution:

$y=\ln(2x+3)$

Differentiate w.r.t. 'x':

$$y_1=\frac{d}{dx}\left[\ln(2x+3)\right]$$

$$y_1=\frac{1}{(2x+3)}\frac{d}{dx}(2x+3)$$

$$\boxed{\because \frac{d}{dx}\ln\left(f(x)\right)=\frac{1}{f(x)}f'(x)}$$

$$y_1=\frac{2}{(2x+3)}$$

$$y_1=2(2x+3)^{-1}$$

Again Differentiate w.r.t. 'x':

$$y_2=\frac{d}{dx}\left[2(2x+3)^{-1}\right]$$

$$y_2=2\left[(-1)(2x+3)^{-2}(2)\right]$$

$$y_2=-4(2x+3)^{-2}$$

$$\boxed{y_2=\frac{-4}{(2x+3)^2}}$$

(iii) $y=x^3e^{2x}$ **12302056**

Solution:

$y=x^3e^{2x}$

$$y_1=\frac{d}{dx}\left(x^3e^{2x}\right)$$

Differentiate w.r.t. 'x':

$$y_1=x^3\frac{d}{dx}\left(e^{2x}\right)+e^{2x}\frac{d}{dx}\left(x^3\right)$$

$$\boxed{\because (uv)'=uv'+vu'}$$

$$y_1=x^3\left(e^{2x}.2\right)+e^{2x}\left(3x^2\right)$$

$$y_1=e^{2x}\left(2x^3\right)+e^{2x}\left(3x^2\right)$$

$$y_1=e^{2x}\left(2x^3+3x^2\right)$$

Again Differentiate w.r.t. 'x':

$$y_2=\frac{d}{dx}\left[e^{2x}\left(2x^3+3x^2\right)\right]$$

$$y_2=e^{2x}\frac{d}{dx}\left(2x^3+3x^2\right)+\left(2x^3+3x^2\right)\frac{d}{dx}\left(e^{2x}\right)$$

$$y_2=e^{2x}\left(6x^2+6x\right)+\left(2x^3+3x^2\right)\left(2e^{2x}\right)$$

$$y_2=e^{2x}\left(6x^2+6x+4x^3+6x^2\right)$$

$$\boxed{y_2=e^{2x}\left(4x^3+12x^2+6x\right)}$$

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(iv) $x^2 - y^2 = 4$ **12302057**

Solution:

$x^2 - y^2 = 4$ .....(i)

Differentiate w.r.t. 'x':

$$\frac{d}{dx}\left(x^2 - y^2\right) = \frac{d}{dx}(4)$$

$$\frac{d}{dx}\left(x^2\right) - \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(4)$$

$$2x - 2yy_1 = 0$$

$$2x = 2yy_1$$

$$y_1 = \frac{2x}{2y}$$

$$y_1 = \frac{x}{y} \text{.....(ii)}$$

Again Differentiate w.r.t. 'x':

$$y_2 = \frac{d}{dx}\left(\frac{x}{y}\right)$$

$$y_2 = \frac{y\frac{d}{dx}(x) - x\frac{d}{dx}(y)}{(y)^2} \qquad \boxed{\because \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}}$$

$$y_2 = \frac{y\cdot 1 - xy_1}{(y)^2}$$

$$y_2 = \frac{y - x\left(\frac{x}{y}\right)}{(y)^2} \qquad \boxed{\because y_1 = \frac{x}{y}}$$

$$y_2 = \frac{y^2 - x^2}{(y)^3}$$

$$y_2 = \frac{-\left(x^2 - y^2\right)}{y^3}$$

By equation (i)

$$\boxed{y_2 = \frac{-4}{y^3}}$$

(v) $x^2 - xy + y^2 = 7$ **12302058**

Solution:

$x^2 - xy + y^2 = 7$ .....(i)

Differentiate w.r.t. 'x':

$$\frac{d}{dx}\left(x^2 - xy + y^2\right) = \frac{d}{dx}(7)$$

$$\frac{d}{dx}\left(x^2\right) - \frac{d}{dx}(xy) + \frac{d}{dx}\left(y^2\right) = 0$$

$$2x - \left(xy_1 + y\cdot 1\right) + 2yy_1 = 0$$

$$2x - xy_1 - y + 2yy_1 = 0$$

$$y_1\left(-x + 2y\right) = y - 2x$$

$$y_1\left(2y - x\right) = y - 2x$$

$$y_1 = \frac{y - 2x}{2y - x} \text{.....(ii)}$$

Again Differentiate w.r.t. 'x':

$$y_2 = \frac{d}{dx}\left(\frac{y - 2x}{2y - x}\right)$$

$$y_2 = \frac{(2y - x)\frac{d}{dx}(y - 2x) - (y - 2x)\frac{d}{dx}(2y - x)}{(2y - x)^2}$$

$$\boxed{\because \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}}$$

$$= \frac{(2y - x)(y_1 - 2) - (y - 2x)(2y_1 - 1)}{(2y - x)^2}$$

$$= \frac{(2yy_1 - 4y - xy_1 + 2x) - (2yy_1 - y - 4xy_1 + 2x)}{(2y - x)^2}$$

$$= \frac{(2yy_1 - 4y - xy_1 + 2x - 2yy_1 + y + 4xy_1 - 2x)}{(2y - x)^2}$$

$$y_2 = \frac{-3y + 3xy_1}{(2y - x)^2}$$

From equation (ii), Putting value of y_1, we have

$$y_2 = \frac{-3y + 3x\left(\frac{y - 2x}{2y - x}\right)}{(2y - x)^2}$$

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$$y_2=\frac{\dfrac{-3y(2y-x)+3x(y-2x)}{(2y-x)}}{(2y-x)^2}$$

$$y_2=\frac{-3y(2y-x)+3x(y-2x)}{(2y-x)^3}$$

$$y_2=\frac{-6y^2+3xy+3xy-6x^2}{(2y-x)^3}$$

$$y_2=\frac{-6y^2+6xy-6x^2}{(2y-x)^3}$$

$$y_2=\frac{-6(x^2-xy+y^2)}{(2y-x)^3}$$

From equation (i), we have

$$y_2=\frac{-6(7)}{(2y-x)^3}$$

$$\boxed{y_2=\frac{-42}{(2y-x)^3}}$$

(vi) $x=a\cos t,\ y=b\sin t$ **12302059**

Solution:

Given:

$x=a\cos t$ .....(i)

Differentiate w.r.t. 'x':

$$\frac{dx}{dt}=-a\sin t$$

Also given

$y=b\sin t$ .....(ii)

Differentiate w.r.t. 't':

$$\frac{dy}{dt}=b\cos t \text{ .....(ii)}$$

By Chain Rule,

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$\frac{dy}{dx}=b\cos t\cdot\frac{1}{-a\sin t}$$

$$\frac{dy}{dx}=-\frac{b\cos t}{a\sin t}$$

$$\frac{dy}{dx}=-\frac{b}{a}\cot t$$

Again Differentiate w.r.t. 'x':

$$\frac{d^2y}{dx^2}=-\frac{b}{a}(-\operatorname{cosec}^2 t)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2}=-\frac{b}{a}(-\operatorname{cosec}^2 t)\left(\frac{1}{-a\sin t}\right)$$

$$\frac{d^2y}{dx^2}=-\frac{b}{a^2}(\operatorname{cosec}^2 t)(\operatorname{cosec} t)$$

$$\frac{d^2y}{dx^2}=-\frac{b}{a^2}\operatorname{cosec}^3 t$$

$$=\frac{-b}{a^2}(\operatorname{cosec} t)^3$$

$$\frac{d^2y}{dx^2}=\frac{-b}{a^2}\left(\frac{b}{y}\right)^3$$

$$\frac{d^2y}{dx^2}=\frac{-b^4}{a^2y^3}$$

$$\boxed{\begin{aligned}&\because\ y=b\sin t\\&\frac{y}{b}=\sin t\\&\frac{b}{y}=\operatorname{cosec} t\end{aligned}}$$

(vii) $x=t^2,\quad y=2t^3$ **12302060**

Solution:

Given that:

$x=t^2$ .....(i)

Differentiate w.r.t 't':

$$\frac{d}{dt}(x)=\frac{d}{dt}(t^2)$$

$$\frac{dx}{dt}=2t$$

Also given

$y=2t^3$

Differentiate w.r.t. 't':

$$\frac{d}{dt}(y)=\frac{d}{dt}(2t^3)$$

$$\frac{dy}{dt}=2(3t^2)=6t^2$$

By Chain Rule,

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$\frac{dy}{dx}=6t^2\cdot\frac{1}{2t}$$

$$\frac{dy}{dx}=3t$$

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Again Differentiate w.r.t. to 'x':

$$\frac{d^2y}{dx^2} = 3\frac{dt}{dx} \qquad \boxed{\because \frac{dt}{dx} = \frac{1}{2t}}$$

$$\frac{d^2y}{dx^2} = 3\left(\frac{1}{2t}\right)$$

$$\frac{d^2y}{dx^2} = \frac{3}{2t} \qquad \boxed{\because x = t^2 \Rightarrow t = \sqrt{x}}$$

$$\frac{d^2y}{dx^2} = \frac{3}{2\sqrt{x}}$$

6. Find y_4 and write expressions for y_{10} and y_{11} without evaluating them:

(i) $y = \sin\left(\frac{x}{2}\right)$ **12302061**

Solution:

Given:

$$y = \sin\left(\frac{x}{2}\right)$$

Differentiate w.r.t. 'x':

$$y_1 = \cos\left(\frac{x}{2}\right)\cdot\frac{1}{2}$$

$$y_2 = \frac{1}{2}\left(-\sin\left(\frac{x}{2}\right)\cdot\frac{1}{2}\right)$$

$$y_2 = -\frac{1}{4}\sin\left(\frac{x}{2}\right)$$

$$y_3 = -\frac{1}{4}\left[\cos\left(\frac{x}{2}\right)\cdot\frac{1}{2}\right]$$

$$y_3 = -\frac{1}{8}\cos\left(\frac{x}{2}\right)$$

$$y_4 = -\frac{1}{8}\left[-\sin\left(\frac{x}{2}\right)\cdot\frac{1}{2}\right]$$

$$y_4 = \frac{1}{16}\sin\left(\frac{x}{2}\right)$$

$$y_4 = \frac{1}{(2)^4}\sin\left(\frac{x}{2}\right)$$

Continuing the same procedure (pattern) we can write the expression for y_{10}, y_{11}

$$\boxed{y_{10} = -\frac{1}{(2)^{10}}\sin\left(\frac{x}{2}\right)}$$

$$\boxed{y_{11} = -\frac{1}{(2)^{11}}\cos\left(\frac{x}{2}\right)}$$

(ii) $y = \ln\left(x^2 - a^2\right)$ **12302062**

Solution:

$$y = \ln\left(x^2 - a^2\right)$$

$$y = \ln(x-a)(x+a)$$

$$\boxed{\because \ln(mn) = \ln m + \ln n}$$

$$y = \ln(x-a) + \ln(x+a)$$

Differentiate w.r.t. 'x':

$$y_1 = \frac{d}{dx}\left[\ln(x-a) + \ln(x+a)\right]$$

$$y_1 = \frac{1}{x-a} + \frac{1}{x+a}$$

$$y_1 = (x-a)^{-1} + (x+a)^{-1}$$

$$y_2 = (-1)(x-a)^{-2} + (-1)(x+a)^{-2}$$

$$y_3 = (2)(x-a)^{-3} + (2)(x+a)^{-3}$$

$$y_4 = -6(x-a)^{-4} - 6(x+a)^{-4}$$

$$y_4 = \frac{-6}{(x-a)^4} - \frac{6}{(x+a)^4}$$

Continuing the same pattern we can write the expression for y_{10}, y_{11}

$$y_{10} = \left[(-1)(-2)\ldots(-9)(x-a)^{-10}\right] + \left[(-1)(-2)\ldots(-9)(x+a)^{-10}\right]$$

$$\boxed{y_{10} = \frac{-9!}{(x-a)^{10}} - \frac{9!}{(x+a)^{10}}}$$

$$y_{11} = \left[(-1)(-2)\ldots(-10)(x-a)^{-11}\right] + \left[(-1)(-2)(-3)\ldots(-10)(x+a)^{-11}\right]$$

$$\boxed{y_{11} = \frac{10!}{(x-a)^{11}} + \frac{10!}{(x+a)^{11}}}$$

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7. If $x^2 + y^2 = 25$, show that 12302063

$$y^3 \frac{d^2y}{dx^2} + 25 = 0.$$

Solution:

Given:

$x^2 + y^2 = 25$ .....(i)

Differentiate w.r.t. 'x':

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = \frac{-x}{y} \text{.....(ii)}$$

Again Differentiate w.r.t. 'x':

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{-x}{y}\right)$$

$$\frac{d^2y}{dx^2} = -\left[\frac{y.1 - x\frac{dy}{dx}}{(y)^2}\right] \qquad \boxed{\because \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}}$$

$$\frac{d^2y}{dx^2} = -\left[\frac{y - x\left(-\frac{x}{y}\right)}{(y)^2}\right]$$

Using equation (ii), we have:

$$\frac{d^2y}{dx^2} = -\left[\frac{\frac{y^2 + x^2}{y}}{y^2}\right]$$

$$\frac{d^2y}{dx^2} = -\left[\frac{y^2 + x^2}{(y)^3}\right]$$

$$y^3 \frac{d^2y}{dx^2} = -\left(x^2 + y^2\right)$$

$$y^3 \frac{d^2y}{dx^2} = -25 \quad \text{[using equation (i)]}$$

$$\boxed{y^3 \frac{d^2y}{dx^2} + 25 = 0}$$

Hence proved.

8. If $y = e^{ax}\cos bx$, show that 12302064

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + \left(a^2 + b^2\right)y = 0$$

Solution:

$y = e^{ax}\cos bx$ .....(i)

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(e^{ax}\cos bx\right)$$

$$\frac{dy}{dx} = e^{ax}\frac{d}{dx}(\cos bx) + \cos bx\frac{d}{dx}\left(e^{ax}\right)$$

$$\boxed{\because (uv)' = uv' + u'v}$$

$$\frac{dy}{dx} = e^{ax}(-\sin bx \cdot b) + \cos bx\left(e^{ax} \cdot a\right)$$

$$\frac{dy}{dx} = -b\left(e^{ax}\sin bx\right) + a\left(e^{ax}\cos bx\right) \text{.....(ii)}$$

Again Differentiate w.r.t. 'x':

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-b\left(e^{ax}\sin bx\right) + a\left(e^{ax}\cos bx\right)\right]$$

$$= -b\frac{d}{dx}\left(e^{ax}\sin bx\right) + a\frac{d}{dx}\left(e^{ax}\cos bx\right)$$

$$\frac{d^2y}{dx^2} = -b\left(e^{ax}\cos bx + \sin bx \cdot e^{ax} \cdot a\right) + a\left[e^{ax}(-\sin bx \cdot b) + \cos bx \cdot e^{ax}(a)\right]$$

$$\frac{d^2y}{dx^2} = -b^2e^{ax}\cos bx - abe^{ax}\sin bx - abe^{ax}\sin bx + a^2e^{ax}\cos bx$$

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$$\frac{d^2y}{dx^2} = a^2e^{ax}\cos bx - 2abe^{ax}\sin bx - b^2e^{ax}\cos bx$$

$$\frac{d^2y}{dx^2} = 2a^2e^{ax}\cos bx - a^2e^{ax}\cos bx - 2abe^{ax}\sin bx - b^2e^{ax}\cos bx$$

$$\frac{d^2y}{dx^2} = 2a^2e^{ax}\cos bx - 2abe^{ax}\sin bx - \left(a^2+b^2\right)e^{ax}\cos bx$$

$$\frac{d^2y}{dx^2} = 2a\left(ae^{ax}\cos bx - be^{ax}\sin bx\right) - \left(a^2+b^2\right)e^{ax}\cos bx$$

$$\frac{d^2y}{dx^2} = 2a\frac{dy}{dx} - \left(a^2+b^2\right)y$$

[using equation (i) and (ii)]

$$\boxed{\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + \left(a^2+b^2\right)y = 0}$$

9. If $x = \sin t$ and $y = \cos^2 t$, show that

$$\frac{d^2y}{dx^2} + 2 = 0$$ **12302065**

Solution:

Given:

$x = \sin t$ .....(i)

Differentiate w.r.t. 't':

$$\frac{d}{dt}(x) = \frac{d}{dt}(\sin t)$$

$$\frac{dx}{dt} = \cos t$$

Also given

$y = \cos^2 t$ .....(ii)

Differentiate w.r.t. 't':

$$\frac{d}{dt}(y) = \frac{d}{dt}\left(\cos^2 t\right)$$

$$\frac{dy}{dt} = -2\cos t\sin t$$

By Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= \frac{-2\sin t\cos t}{\cos t}$$

$$\boxed{\frac{dy}{dx} = -2\sin t}$$

Again Differentiate w.r.t. 'x':

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-2\sin t)$$

$$\frac{d^2y}{dx^2} = -2(\cos t)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = -2(\cos t)\left(\frac{1}{\cos t}\right) \qquad \boxed{\because \frac{dt}{dx} = \frac{1}{\cos t}}$$

$$\frac{d^2y}{dx^2} = -2$$

$$\boxed{\frac{d^2y}{dx^2} + 2 = 0}$$

Hence proved.

10. If $x = \ln t$ and $y = \frac{1}{t}$, show that **12302066**

$$\frac{d^2y}{dx^2} - y = 0$$

Solution:

Given:

$x = \ln t$ .....(i)

Differentiate w.r.t. 't':

$$\frac{d}{dt}(x) = \frac{d}{dt}(\ln t)$$

$$\frac{dx}{dt} = \frac{1}{t}$$

Also Given:

$$y = \frac{1}{t}$$.....(ii)

$$y = t^{-1}$$

Differentiate w.r.t. 't':

$$\frac{d}{dt}(y) = \frac{d}{dt}\left(t^{-1}\right)$$

$$\frac{dy}{dt} = \frac{-1}{t^2}$$

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By Chain Rule

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$$
$$=-\frac{1}{t^2}\cdot t$$
$$=-\frac{1}{t}$$
$$\frac{dy}{dx}=-(t)^{-1}$$

Again Differentiate w.r.t. 'x':

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left[(t)^{-1}\right]$$
$$\frac{d^2y}{dx^2}=-(-1)(t)^{-2}\frac{dt}{dx}$$
$$\frac{d^2y}{dx^2}=\frac{1}{(t)^2}(t)$$
$$\frac{d^2y}{dx^2}=\frac{1}{t}$$
$$\frac{d^2y}{dx^2}=y \qquad \text{(given)}$$
$$\boxed{\frac{d^2y}{dx^2}-y=0}$$

Hence proved.

11. If $y=a\cos(\ln x)+b\sin(\ln x)$, show that $x^2y_2+xy_1+y=0$ **12302067**

Solution:

$$y=a\cos(\ln x)+b\sin(\ln x) \quad \ldots\ldots\ldots(i)$$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y)=\frac{d}{dx}\left[a\cos(\ln x)+b\sin(\ln x)\right]$$
$$y_1=\frac{dy}{dx}=-a\sin(\ln x)\cdot\frac{1}{x}+b\cos(\ln x).\frac{1}{x}$$
$$y_1=\frac{1}{x}\left[-a\sin(\ln x)+b\cos(\ln x)\right]$$
$$xy_1=-a\sin(\ln x)+b\cos(\ln x)$$

Again Differentiate w.r.t. 'x':

$$\frac{d}{dx}(xy_1)=\frac{d}{dx}\left[-a\sin(\ln x)+b\cos(\ln x)\right]$$
$$xy_2+y_1\cdot 1=-a\cos(\ln x)\cdot\frac{1}{x}-b\sin(\ln x)\cdot\frac{1}{x}$$
$$xy_2+y_1=\frac{1}{x}\left[-a\cos(\ln x)-b\sin(\ln x)\right]$$
$$x(xy_2+y_1)=-\left[a\cos(\ln x)+b\sin(\ln x)\right]$$
$$x^2y_2+xy_1=-y \text{ [Using equation (i)]}$$
$$\boxed{x^2y_2+xy_1+y=0}$$

Hence proved.

Increasing and Decreasing Functions

Increasing Function

Let I be an interval and let $f:I\to\mathbb{R}$ be a function. We say that f is increasing if for any $x_1,x_2\in I$.

$$x_1<x_2\Rightarrow f(x_1)\le f(x_2)$$

Decreasing Function

Let I be an interval and let $f:I\to\mathbb{R}$ be a function. We say that f is decreasing if for any $x_1,x_2\in I$.

$$x_1<x_2\Rightarrow f(x_1)\ge f(x_2)$$

Recall!

$(a,b),[a,b],(a,b],[a,b),(-\infty,a),(-\infty,a],$ $(a,\infty),[a,\infty)$ all are intervals.

Let f be a differentiable function defined on an interval I. Then

(i) f is increasing on I, if $f'(x)\ge 0$ for each $x\in I$.

(ii) f is decreasing on I, if $f'(x)\le 0$ for each $x\in I$.

Example 15: Prove that $f(x)=e^x$ defined on $R=(-\infty,\infty)$ is increasing. **12302068**

Solution: We differentiate $f(x)=e^x$ with respect to 'x' to obtain.

$f'(x)=e^x>0$ for each $x\in(-\infty,\infty)$

Hence, $f(x)=e^x$ is an increasing function on $R=(-\infty,\infty)$.

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Example 16: **Show that $f(x)=2^{-x}$ defined on $R=(-\infty,\infty)$ is decreasing.** 12302069

Solution: We differentiate $f(x)=2^{-x}$ with respect to 'x' to obtain.

$$f'(x)=2^{-x}\frac{d}{dx}(-x)$$

$f'(x)=-2^{-x}<0$ for each $x\in(-\infty,\infty)$

So, $f(x)=2^{-x}$ is decreasing function on $R=(-\infty,\infty)$

Stationary Point

A point on the graph where its derivative is zero is called a stationary point.

Example 17: **Find the intervals on which the function $f(x)=|x|$, defined on $\mathbb{R}=(-\infty,\infty)$ is increasing or decreasing.**

Solution: We know that: 12302070

$$f(x)=|x|=\begin{cases}x \text{ if } x\geq 0\\ -x \text{ if } x<0\end{cases}$$

Differentiating w.r.t. 'x':

$$f'(x)=\begin{cases}1 \text{ if } x\geq 0\\ -1 \text{ if } x<0\end{cases}$$

We now evaluate $f'(0)$ by definition:

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

(By definition of derivative)

$$f'(0)=\lim_{h\to 0}\frac{f(h)-0}{h}=\lim_{h\to 0}\frac{|h|}{h} \text{ and}$$

$$\lim_{h\to 0^-}\frac{|h|}{h}=\lim_{h\to 0^-}\frac{-h}{h}=-1 \text{ and } \lim_{h\to 0^+}\frac{|h|}{h}=\lim_{h\to 0^+}\frac{h}{h}=1$$

Since $f(x)=|x|$ is defined different on left and right of $x=0$, we compute one sided limits.

Since left-hand and right-hand limits are different, the limit $\lim_{h\to 0}\frac{|h|}{h}$ does not exist.

Hence $f'(0)$ is not defined.

($\because$ x is a critical point but not stationary)

The function has no stationary point because there is no point where the derivative is equal to zero.

Since $f'(x)=-1<0$ for each $x\in(-\infty,0)\Rightarrow$ the function is decreasing on $(-\infty,0)$ and $f'(x)=1>0$ for each $x\in(0,\infty)$ the function is increasing on $(0,\infty)$.

Hence $f(x)=|x|$ is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$.

Conclusion: From these examples, we conclude that a function may change its:

1. Increasing/decreasing behavior around:
   - Stationary points where $f'(x)=0$, or
   - Points where $f'(x)$ is undefined.

Critical Point

For a function $f(x)$ a point $x=c, c\in$ Dom f is called a critical point if:

$f'(c)=0$ or $f'(c)$ does not exist.

Examples:

1. For $f(x)=|x|$, the derivative does not exist at $x=0$, so $x=0$ is a critical point.
2. For $f(x)=x^2$, the derivative $f'(x)=2x$ is zero at $x=0$, so $x=0$ is a critical point.

Method to Find Intervals Where a Function is Increasing or Decreasing

To find the intervals where a function is increasing or decreasing, follow these steps:

Step 1: Differentiate the given function $f(x)$ to find $f'(x)$.

Step 2: Find critical points by solving $f'(x)=0$ and identifying points where $f'(x)$ does not exist.

Step 3: If critical points exist, divide the domain into sub-intervals using the critical points.

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Step 4: Select a test points exist, from each sub-interval and substitute into $f'(x)$:

- If $f'(x)>0$ in a sub-interval, the function is increasing there.
- If $f'(x)<0$ in a sub-interval, the function is decreasing there.

Example 18: Find the intervals in which the function $f(x)=2x^3-3x^2-12x$ is increasing or decreasing. 12302071

Solution:

The function $f(x)=2x^3-3x^2-12x$ is defined for all $x\in R=(-\infty,\infty)$.

We differentiate $f(x)=2x^3-3x^2-12x$ with respect to x to obtain:

$$f'(x)=6x^2-6x-12 \quad \ldots\ldots\ldots(i)$$

It is clear from equation (i) that $f'(x)$ is defined for each $x\in(-\infty,\infty)$.

To find other critical points, we solve $f'(x)=0$:

$$6x^2-6x-12=0$$
$$6(x^2-x-2)0$$
$$6(x+1)(x-2)=0$$
$$x=-1 \text{ and } \quad x=2$$

Hence $x=-1$ and $x=2$ are the critical points of the given function.

Mark these points on the real line. These critical points divide the domain into intervals $(-\infty,-1),(-1,2)$ and $(2,\infty)$.



Now we check the sing of $f'(x)$ in each subinterval.

Interval	Test point	Sign of	Conclusion
$(-\infty,-1)$	$x=-2$	$f'(-2)=6(-2)^2-6(-2)-12=24>0$	Increasing
$(-1,2)$	$x=0$	$f'(0)=6(0)^2-6(0)-12=-12<0$	Decreasing
$(2,\infty)$	$x=3$	$f'(3)=6(3)^2-6(3)-12=24>0$	Increasing

Hence the function $f(x)=2x^3-3x^2-12x$:

- Increases on $(-\infty,-1)$ and $(2,\infty)$.
- Decreases on $(-1,2)$.

Example 19: Find the intervals where the function $f(x)=x^{\frac{2}{3}}$ is increasing or decreasing on the domain $x\in(-\infty,\infty)$.

Solution: 12302072

We differentiate $f(x)=x^{\frac{2}{3}}$ with respect to 'x' to obtain:

$$f'(x)=\frac{2}{3}x^{\frac{-1}{3}}=\frac{2}{3x^{\frac{1}{3}}} \quad \ldots\ldots\ldots(i)$$

From equation (i), we conclude that $f'(0)$ is not defined. Thus $x=0$ is a critical point of the function $f'(x)=x^{\frac{2}{3}}$.

To find other critical points we solve $f'(x)=0$.

$$\frac{2}{3x^{\frac{1}{3}}}=0$$

The above equation has no solution. Thus, the only critical point is $x=0$. This critical point divides the domain in the following intervals. $(-\infty,0)$ and $(0,\infty)$.

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Choose $x=-1\in(-\infty,0)$ and substitute into equation (i).

$$f'(-1)=\frac{2}{3(-1)^{\frac{1}{3}}}=-\frac{2}{3}<0\Rightarrow f'(x)<0 \text{ for}$$

all $x\in(-\infty,0)$ $\left(\because(-1)^{\frac{1}{3}}=-1\right)$

Thus, the function is decreasing on $(-\infty,0)$.

Choose $x=1\in(-\infty,0)$ and substitute into equation (i).

$$f'(1)=\frac{2}{3(1)^{\frac{1}{3}}}=\frac{2}{3}>0\Rightarrow f'(x)>0 \quad \text{for all}$$

$x\in(0,\infty)$.

Thus, the function is increasing on $(0,\infty)$.

Hence the function $f(x)=x^{\frac{2}{3}}$ defined on $(-\infty,\infty)$:

- Decreases on $(-\infty,0)$
- Increases on $(0,\infty)$

Relative and Absolute Extrema

Global (Absolute) Maximum

Let I be an interval and let $f:I\to\mathbb{R}$ be a function. We say that f has a global (absolute) maximum at $x=c\in I$, if $f(c)\geq f(x)$ for all $x\in I$.

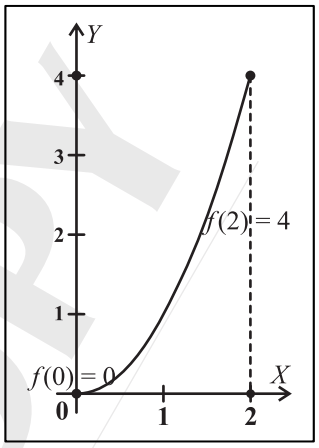
If $f(x)$ has a global (absolute) maximum at $x=c$, then $f(c)$ is called global (absolute) maximum value of the function.

Global (Absolute) Minimum

Let I be an interval and let $f:I\to\mathbb{R}$ be a function. We say that f has a global (absolute) minimum at $x=c\in I$, if $f(c)\leq f(x)$ for all $x\in I$.

The global maximum and minimum of a function are collectively called its global extrema.

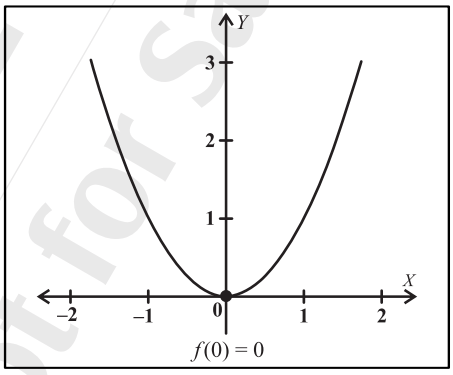
Example: (a) Let $f(x)=x^2$ be defined on the interval $[0,2]$. The function f has a global maximum at $x=2$ and a global minimum at $x=0$. The corresponding maximum and minimum values of the function are:



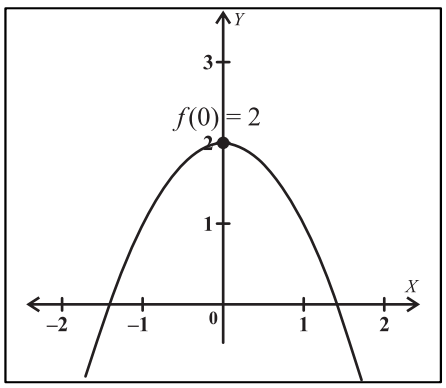
$f(2)=4$ (global maximum value)

$f(0)=0$ (global minimum value)

(b) The function $f(x)=x^2$ defined on $(-\infty,\infty)$ has an absolute minimum value at $x=0$, where $f(0)=0$. It has no absolute maximum value.

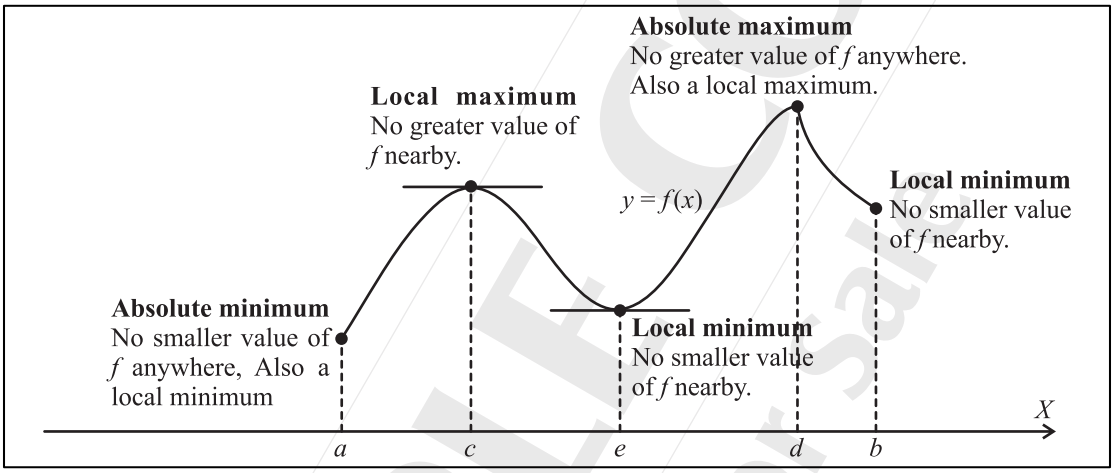


(c) The function $f(x)=-x^2+2$ defined on $(-\infty,\infty)$ has an absolute maximum value at $x=0$, where $f(0)=2$. It has no absolute minimum value.



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Local (Relative) Maxima	Local (Relative) Minima
Let I be an interval and let $f: I \to \mathbb{R}$ be a function. We say that f has a local (relative) maxima at $x = c \in I$, if there exists an open interval I_1 such that $c \in I_1$ and: $f(c) \geq f(x)$, for all $x \in I \cap I_1$ If $f(x)$ has a local (relative) maximum at $x = c$, then $f(c)$ is called local (relative) maximum value of the function	Let I be an interval and let $f: I \to \mathbb{R}$ be a function. We say that f has a local (relative) minima at $x = e \in I$, if there exists an open interval I_1 such that $e \in I_1$ and: $f(e) \leq f(x)$ for all $x \in I \cap I_1$ If $f(x)$ has a local (relative) minimum at $x = e$, then $f(e)$ is called local (relative) minimum value of the function.



Method to Find Global Extrema of a Continuous Functions Defined on a Closed Interval [*a*, *b*]

1. Find all critical point of the function within the interval [*a*, *b*].
2. Find the values of the function at critical points and at end points of the interval.
3. The largest of these values is the global maximum value, and the smallest is the global minimum value.

Example 20: Find the global extreme values of $f(x) = x^4 - 4x$ on $[-2, 3]$. **12302073**

Solution:

We are given: $f(x) = x^4 - 4x$ .....(i)

We differentiate equation (i) with respect to 'x' to get:

$f'(x) = 4x^3 - 4$ .....(ii)

From equation (ii) it is clear that $f'(x)$ exists on $[-2, 3]$.

For stationary points, we solve $f'(x) = 0$:

$4x^3 - 4 = 0$

$4(x^3 - 1) = 0$

$(x-1)(x^2 + x + 1) = 0$

$\because a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Either: $x - 1 = 0$ or $x^2 + x + 1 = 0$

The equation $x^2 + x + 1 = 0$ has complex roots, so the only real stationary point is $x = 1$.

Critical Point Value: $f(1) = 1^4 - 4(1) = -3$

End Point Value:

$f(-2) = (-2)^4 - 4(-2) = 16 + 8 = 24$

$f(3) = (3)^4 - 4(3) = 81 - 12 = 69$

From the values $-3, 24, 69$:

- The global maximum is $f(3) = 69$
- The global minimum is $f(1) = -3$

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Second Derivative Test

Let f be a twice differentiable function at $x=c$, where c lies in the domain of f. Suppose that $f'(c)=0$. Then:

1. f has relative maximum at c if $f^n(c)<0$.
2. f has relative minimum at c, if $f^n(c)>0$.
3. If $f^n(c)=0$, then the test in inclusive, and the nature of the critical point cannot be determined using the second derivative test alone.

Point to Note

If first derivative is not defined at a critical point, can we apply second derivative test at that critical point.

Example 21: Examine the function $f(x)=\frac{1}{2}x^4-\frac{4}{3}x^3-x^2+4x+1$ for relative extreme values. 12302074

Solution:

We are given:

$$f(x)=\frac{1}{2}x^4-\frac{4}{3}x^3-x^2+4x+1 \ldots\ldots\ldots\text{(i)}$$

Differentiate w.r.t. 'x':

$$f'(x)=2x^3-4x^2-2x+4 \ldots\ldots\ldots\text{(ii)}$$

Set $f'(x)=0$:

$$2x^3-4x^2-2x+4=0$$
$$2\left(x^3-2x^2-x+2\right)=0$$
$$x^3-2x^2-x+2=0$$
$$x^3-x-2x^2+2=0$$
$$x\left(x^2-1\right)-2\left(x^2-1\right)=0$$
$$(x-2)\left(x^2-1\right)=0$$
$$(x-2)(x-1)(x+1)=0$$

Thus, the critical points are:

$x=-1, x=1, x=2$

We differentiate the function $f'(x)=2x^3-4x^2-2x+4$ w.r.t. 'x',

$$f''(x)=6x^2-8x-2$$
$$f''(x)=2\left(3x^2-4x-1\right) \ldots\ldots\ldots\text{(iii)}$$

Substitute $x=-1$ into equation (iii):

$$f''(-1)=2\left[3(-1)^2-4(-1)-1\right]=12>0$$

Thus, the function has a relative minimum at $x=-1$.

Substitute $x=1$ into equation (iii):

$$f''(x)=2\left[3(1)^2-4(1)-1\right]=-4<0$$

Thus, the function has a relative maximum at $x=1$.

Substitute $x=2$ into equation (iii):

$$f''(2)=2\left[3(2)^2-4(2)-1\right]=6>0$$

Thus, the function has a relative minimum at $x=2$.

Substitute $x=-1, x=1$ and $x=2$ into equation (i):

$$f(-1)=\frac{1}{2}(-1)^4-\frac{4}{3}(-1)^3-(-1)^2+4(-1)+1$$
$$=\frac{1}{2}+\frac{4}{3}-4=-\frac{13}{6}$$

$$f(1)=\frac{1}{2}(1)^4-\frac{4}{3}(1)^3-(1)^2+4(1)+1$$
$$=-\frac{5}{6}+4=\frac{19}{6}$$

$$f(2)=\frac{1}{2}(2)^4-\frac{4}{3}(2)^3-(2)^2+4(2)+1=\frac{7}{3}$$

Finally,

- The relative minimum value at $x=-1$ is $f(-1)=-\frac{13}{6}$.
- The relative maximum value at $x=1$ is $f(1)=\frac{19}{6}$.
- The relative minimum value at $x=2$ is $f(2)=\frac{7}{3}$.

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EXERCISE 2.4

1. Determine the intervals on which is increasing or decreasing for the domain mentioned in each case.

(i) $f(x)=\sin x \;\; ; \;\; x\in(0,2\pi)$ **12302075**

Solution:

$f(x)=\sin x \quad x\in(0,2\pi)$

Differentiate w.r.t. 'x':

$$f'(x)=\frac{d}{dx}(\sin x)$$

$$f'(x)=\cos x$$

For stationary points:

Put $f'(x)=0$

$$\cos x=0$$

$$x=\frac{\pi}{2},\frac{3\pi}{2}$$

So, sub intervals are $\left(0,\frac{\pi}{2}\right)$, $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$, $\left(\frac{3\pi}{2},2\pi\right)$.

Test point: $x=\frac{\pi}{4}\in\left(0,\frac{\pi}{2}\right)$

$f'\left(\frac{\pi}{4}\right)=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}>0$ (Increasing)

Test point: $x=\pi\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$

$f'(\pi)=\cos\pi=-1<0$ (Decreasing)

Now, **Test point:** $x=\frac{7\pi}{4}\in\left(\frac{3\pi}{2},2\pi\right)$

$f'\left(\frac{7\pi}{4}\right)=\cos\left(\frac{7\pi}{4}\right)=\frac{1}{\sqrt{2}}>0$ (Increasing)

Finally:

- Increasing on $\left(0,\frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2},2\pi\right)$
- Decreasing on $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$

(ii) $f(x)=4-x^2 \;\; ; \;\; x\in\mathbb{R}=(-\infty,\infty)$

Solution: **12302076**

$f(x)=4-x^2 \;\; ; \;\; x\in\mathbb{R}=(-\infty,\infty)$

$f(x)=4-x^2 \quad x\in R$

Differentiate w.r.t. 'x':

$$f'(x)=\frac{d}{dx}(4-x^2)$$

$$f'(x)=-2x$$

For stationary points:

Put $f'(x)=0$

$$-2x=0$$

$$x=0$$

Intervals are $(-\infty,0)$ and $(0,\infty)$.

Test point: $x=-1\in(-\infty,0)$

$f'(-1)=-2(-1)=2>0$ (Increasing)

Test point: $x=1\in(0,\infty)$

$f'(1)=-2(1)=-2<0$ (Decreasing)

Finally:

- Increasing on $(-\infty,0)$
- Decreasing on $(0,\infty)$

(iii) $f(x)=x^2+3x+2 \;\; ; \;\; x\in[-4,1]$ **12302077**

Solution:

$f(x)=x^2+3x+2 \;\; ; \;\; x\in[-4,1]$

Differentiate w.r.t. 'x':

$$f'(x)=\frac{d}{dx}(x^2+3x+2)$$

$$f'(x)=2x+3$$

For stationary points:

Put $f'(x)=0$

$$2x+3=0$$

$$2x=-3$$

$$x=\frac{-3}{2}$$

So, intervals are $\left(-4,\frac{-3}{2}\right),\left(\frac{-3}{2},1\right)$

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Test point: $x=-2\in\left[\left(-4,\frac{-3}{2}\right)\right]$

$f'(-2)=2(-2)+3=-1<0$ (Decreasing)

Test point: $x=0\in\left(\frac{-3}{2},1\right)$

$f'(0)=2(0)+3>0$ (Increasing)

Finally:

- Increasing on $\left(\frac{-3}{2},1\right)$
- Decreasing on $\left(-4,\frac{-3}{2}\right)$

(iv) $f(x)=\sin x+\frac{1}{4}\cos 2x \quad ; \quad x\in[0,2\pi]$

Solution: 12302078

$f(x)=\sin x+\frac{1}{4}\cos 2x \quad ; \quad x\in[0,2\pi]$

Differentiate w.r.t. 'x':

$$f'(x)=\frac{d}{dx}\left[\sin x+\frac{1}{4}\cos 2x\right]$$

$$f'(x)=\cos x+\frac{1}{4}(-\sin 2x\cdot 2)$$

$$f'(x)=\cos x-\frac{2}{4}\sin 2x$$

$$f'(x)=\cos x-\frac{1}{2}\sin 2x$$

For stationary points:

Put $f'(x)=0$

$$\cos x-\frac{1}{2}\sin 2x=0$$

$$\cos x=\frac{1}{2}\sin 2x$$

$$\cos x=\frac{1}{2}(2\sin x\cos x)$$

$$\cos x-\sin x\cos x=0$$

$$\cos x(1-\sin x)=0$$

$\cos x=0 \quad , \quad 1-\sin x=0$

$\cos x=0 \quad , \quad \sin x=1$

$x=\frac{\pi}{2},\frac{3\pi}{2} \quad , \quad x=\frac{\pi}{2}$

So, intervals are $\left[\left(0,\frac{\pi}{2}\right)\left(\frac{\pi}{2},\frac{3\pi}{2}\right)\left(\frac{3\pi}{2},2\pi\right)\right]$

Test point: $x=\frac{\pi}{4}\in\left[\left(0,\frac{\pi}{2}\right)\right]$

$$f'\left(\frac{\pi}{4}\right)=\cos\frac{\pi}{4}-\frac{1}{2}\sin 2\left(\frac{\pi}{4}\right)$$

$$=\cos\frac{\pi}{4}-\frac{1}{2}\sin\frac{\pi}{2}$$

$f'\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}-\frac{1}{2}>0$ (Increasing)

Test point: $x=\pi\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$

$$f'(\pi)=\cos\pi-\frac{1}{2}\sin 2\pi$$

$$f'(\pi)=-1-\frac{1}{2}(0)$$

$f'(\pi)=-1<0$ (Decreasing)

Test point: $x=\frac{7\pi}{4}\in\left[\left(\frac{3\pi}{2},2\pi\right)\right]$

$$f'\left(\frac{7\pi}{4}\right)=\cos\frac{7\pi}{4}-\frac{1}{2}\sin 2\left(\frac{7\pi}{4}\right)$$

$$f'\left(\frac{7\pi}{4}\right)=\cos\frac{7\pi}{4}-\frac{1}{2}\sin\frac{7\pi}{2}$$

$$f'\left(\frac{7\pi}{4}\right)=\frac{1}{\sqrt{2}}-\frac{1}{2}\left(-\frac{1}{\sqrt{2}}\right)$$

$f'\left(\frac{7\pi}{4}\right)=\frac{1}{\sqrt{2}}+\frac{1}{2\sqrt{2}}>0$ (Increasing)

- Increasing on $\left[\left(0,\frac{\pi}{2}\right)\right]$ and $\left[\left(\frac{3\pi}{2},2\pi\right)\right]$
- Decreasing on $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$

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2. Using the Second Derivative Test, find local extreme values of the following functions:

(i) $f(x)=8-2x^2$ **12302079**

Solution:

$f(x)=8-2x^2$

Differentiate w.r.t. 'x':

$$f'(x)=\frac{d}{dx}\left(8-x^2\right)$$

$$f'(x)=\frac{d}{dx}(8)-\frac{d}{dx}\left(x^2\right)$$

$$f'(x)=0-4x$$

$$f'(x)=-4x$$

For stationary points:

$$f'(x)=0$$

$$-4x=0$$

$$x=0$$

Differentiate w.r.t. 'x'

$$f''(x)=\frac{d}{dx}(-4x)$$

$$f''(x)=-4<0$$

Local Maxima at $x=0$

Put $x=0$ in $f(x)$

$$f(0)=8-2(0)^2$$

$$f(0)=8$$

The function has local Maxima at $x=0$ is $f(0)=8$.

(ii) $f(x)=x^2-x-2$ **12302080**

Solution:

$f(x)=x^2-x-2$

Differentiate w.r.t. 'x':

$$f'(x)=\frac{d}{dx}\left(x^2-x-2\right)$$

$$=\frac{d}{dx}\left(x^2\right)-\frac{d}{dx}(x)-\frac{d}{dx}(2)$$

$$=2x-1-0$$

$$f'(x)=2x-1$$

For stationary points:

Put $f'(x)=0$

$$\Rightarrow 2x-1=0$$

$$2x=1$$

$$x=\frac{1}{2}$$

$x=\frac{1}{2}$ is the critical point.

Now, $f'(x)=2x-1$

Differentiate w.r.t. 'x':

$f''(x)=2>0$

That function has local minima at $x=\frac{1}{2}$.

Now,

$$f\left(\frac{1}{2}\right)=\left(\frac{1}{4}\right)-\frac{1}{2}-2$$

$$=\frac{1-2-8}{4}$$

$$f\left(\frac{1}{2}\right)=f\left(\frac{1}{2}\right)=\frac{-9}{4}$$

Finally: The relative minimum value at $x=\frac{1}{2}$ is $\frac{-9}{4}$.

(iii) $f(x)=x^4-8x^2+2$ **12302081**

Solution:

$f(x)=x^4-8x^2+2$ .....(i)

Differentiate w.r.t. 'x':

$$f'(x)=\frac{d}{dx}\left(x^4-8x^2+2\right)$$

$f'(x)=4x^3-16x$ .....(ii)

For stationary points:

$$f'(x)=0$$

$$4x^3-16x=0$$

$$4x\left(x^2-4\right)=0$$

$$4x(x+2)(x-2)=0$$

$$4x=0,\ x+2=0,\ x-2=0$$

$$x=0,\quad x=-2,\quad x=2$$

which are critical points.

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Now, $f'(x)=4x^3-16x$

Differentiate w.r.t. 'x':

$f''(x)=12x^2-16$ .....(iii)

Substitute $x=0$ in equation (iii)

$f''(0)=12(0)^2-16=-16<0$

Thus, function has Relative Maxima at $x=0$.

Substitute $x=-2$ in equation (iii)

$f''(-2)=12(-2)^2-16=48-16=32>0$

Thus, $f(x)$ has relative minima at $x=-2$.

Substitute $x=2$ in equation (iii)

$f''(2)=12(2)^2-16=48-16=32>0$

So, $f(x)$ has relative minima at $x=2$.

Substitute $x=0,2,-2$ in equation (i)

$f(0)=(0)^4-8(0)^2+2=2$

$f(2)=(2)^4-8(2)^2+2$

$=16-32+2=-14$

And

$f(-2)=(-2)^4-8(-2)^2+2$

$=16-32+2=-14$

Finally,

The relative maximum value at $x=0$ is $f(0)=-16$

The relative minimum value at $x=\pm 2$ is $f(\pm 2)=-14$.

(iv) $f(x)=(x+2)^2(x-3)$ **12302082**

Solution:

$f(x)=(x+2)^2(x-3)$.....(i)

Differentiate w.r.t. 'x':

$f'(x)=(x-3)\frac{d}{dx}(x+2)^2+(x+2)^2\frac{d}{dx}(x-3)$

$f'(x)=(x-3)\cdot 2(x+2)+(x+2)^2(1-0)$

$=2(x-3)(x+2)+(x+2)^2$

$=(x+2)[2(x-3)+(x+2)]$

$=(x+2)(2x-6+x+2)$

$f'(x)=(x+2)(3x-4)$.....(ii)

For stationary points:

Put $f'(x)=0$

$(x+2)(3x-4)=0$

$x+2=0,\ 3x-4=0$

$x=-2, \qquad x=\frac{4}{3}$

So, $x=-2,\frac{4}{3}$ are critical points of the function.

Now,

Again differentiate w.r.t. 'x':

$f''(x)=\frac{d}{dx}[(x+2)(3x-4)]$

$=(x+2)\frac{d}{dx}(3x-4)+(3x-4)\frac{d}{dx}(x+2)$

$=(x+2)(3-0)+(3x-4)(1)$

$=3(x+2)+3x-4$

$=3x+6+3x-4$

$f''(x)=6x+2$ .....(iii)

Substitute $x=-2$ in equation (iii)

$f''(-2)=6(-2)+2=-12+2=-10<0$

Thus function has relative maxima at $x=-2$

Substitute $x=\frac{4}{3}$ in equation (iii)

$f''\left(\frac{4}{3}\right)=6\left(\frac{4}{3}\right)+2=8+2=10>0$

Thus, function has relative minima at $x=\frac{4}{3}$.

Now put $x=-2,\frac{4}{3}$ in equation (i)

$f(-2)=(-2+2)^2(-2-3)=(0)^2(-5)=0$

$f\left(\frac{4}{3}\right)=\left(\frac{4}{3}+2\right)^2\left(\frac{4}{3}-3\right)$

$=\left(\frac{4+6}{3}\right)^2\left(\frac{4-9}{3}\right)=\left(\frac{10}{3}\right)^2\left(\frac{-5}{3}\right)$

$=\frac{100}{9}\times\frac{-5}{3}$

$f\left(\frac{4}{3}\right)=\frac{-500}{27}$

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Finally:

- The relative maximum value at $x=-2$ is $f(-2)=0$
- The relative minimum value at $x=\frac{4}{3}$ is $f\left(\frac{4}{3}\right)=\frac{-500}{27}$.

(v) $f(x)=\left(x^2-2x\right)e^x$ **12302083**

Solution:

$f(x)=\left(x^2-2x\right)e^x$ .....(i)

Differentiate w.r.t. 'x':

$$f'(x)=\frac{d}{dx}\left[\left(x^2-2x\right)e^x\right]$$
$$=\left(x^2-2x\right)\frac{d}{dx}e^x+e^x\frac{d}{dx}\left(x^2-2x\right)$$
$$=\left(x^2-2x\right)e^x+e^x\left(2x-2\right)$$
$$f'(x)=x^2e^x-2xe^x+2xe^x-2e^x$$
$$f'(x)=x^2e^x-2e^x$$
$$f'(x)=e^x\left(x^2-2\right) \text{.....(ii)}$$

For stationary points:

Put $e^x\left(x^2-2\right)=0$

$e^x\neq 0,\ x^2-2=0$

$x^2=2$

$x=\pm\sqrt{2}$

$x=+\sqrt{2},-\sqrt{2}$

which are critical points of the function.

Now, $f'(x)=e^x\left(x^2-2\right)$

Differentiate w.r.t. 'x':

$$f''(x)=\frac{d}{dx}\left[e^x\left(x^2-2\right)\right]$$
$$=e^x\frac{d}{dx}\left(x^2-2\right)+\left(x^2-2\right)\frac{d}{dx}e^x$$
$$=e^x\left(2x-0\right)+\left(x^2-2\right)e^x$$
$$=2xe^x+x^2e^x-2e^x$$
$$f''(x)=e^x\left(x^2+2x-2\right) \text{.....(iii)}$$

Substitute $x=\sqrt{2}$ in equation (iii)

$$f''\left(\sqrt{2}\right)=e^{\sqrt{2}}\left[\left(\sqrt{2}\right)^2+2\sqrt{2}-2\right]$$
$$=e^{\sqrt{2}}\left(2+2\sqrt{2}-2\right)$$
$$=2\sqrt{2}e^{\sqrt{2}}>0$$

Thus, function has relative minima at $x=\sqrt{2}$.

Substitute $x=-\sqrt{2}$ in equation (ii),

$$f''\left(-\sqrt{2}\right)=e^{-\sqrt{2}}\left[\left(-\sqrt{2}\right)^2+2\left(-\sqrt{2}\right)-2\right]$$
$$=e^{-\sqrt{2}}\left(2-2\sqrt{2}-2\right)$$
$$=e^{-\sqrt{2}}\left(-2\sqrt{2}\right)$$
$$=-2\sqrt{2}e^{-\sqrt{2}}<0$$

The function has relative maxima at $x=-\sqrt{2}$

Substitute $x=\sqrt{2},-\sqrt{2}$ in equation (i)

$$f\left(\sqrt{2}\right)=\left[\left(\sqrt{2}\right)^2-2\sqrt{2}\right]e^{\sqrt{2}}$$
$$=\left(2-2\sqrt{2}\right)e^{\sqrt{2}}$$
$$f\left(-\sqrt{2}\right)=\left[\left(-\sqrt{2}\right)^2-2\sqrt{2}\right]e^{-\sqrt{2}}$$
$$f\left(-\sqrt{2}\right)=\left(2+2\sqrt{2}\right)e^{-\sqrt{2}}$$

Finally:

- The relative maximum value at $x=-\sqrt{2}$ is $f\left(-\sqrt{2}\right)=\left(2+2\sqrt{2}\right)e^{-\sqrt{2}}$
- The relative minimum value at $x=\sqrt{2}$ is $f\left(\sqrt{2}\right)=\left(2-\sqrt{2}\right)e^{\sqrt{2}}$.

3. Discuss the global extrema of the following functions over specified domains.

(i) $f(x)=\sin x+\cos x$ on $[0,2\pi]$ **12302084**

Solution:

$f(x)=\sin x+\cos x$ .....(i)

Differentiate w.r.t. 'x':

$$f'(x)=\frac{d}{dx}[\sin x+\cos x]$$
$$=\frac{d}{dx}(\sin x)+\frac{d}{dx}(\cos x)$$

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$f'(x) = \cos x - \sin x$

For stationary points:

$f'(x) = 0$

$\cos - \sin x = 0$

$\cos x = \sin x$

$\frac{\cos x}{\cos x} = \frac{\sin x}{\cos x}$

$\tan x = 1$

$x = \frac{\pi}{4}, \quad \frac{5\pi}{4}$

which are critical points.

End point values:

Putting in equation (i)

$f(0) = \sin(0) + \cos(0)$

$f(0) = 1$

$f(2\pi) = \sin 2\pi + \cos 2\pi$

$f(2\pi) = 1$

Critical point values:

Putting in equation (i)

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$f\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{4} + \cos\frac{5\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}}$$

$f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$

From the values: $1, \sqrt{2}, -\sqrt{2}$

- The Global Maximum is $f\left(\frac{\pi}{4}\right) = \sqrt{2}$
- The Global Minimum is $f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$

(ii) $f(x) = x^{4/3}$ on $[-1, 8]$ **12302085**

Solution:

$f(x) = x^{4/3}$ on $[-1, 8]$

$f(x) = x^{\frac{4}{3}}$ .....(i)

Differentiate w.r.t. 'x':

$f'(x) = \frac{d}{dx}\left(x^{3/4}\right)$

$= \frac{3}{4}x^{\frac{3}{4}-1}$

$f'(x) = \frac{4}{3}x^{1/3}$

For stationary points:

Put $f'(x) = 0$

$\frac{4}{3}(x)^{1/3} = 0$

$x^{1/3} = 0$

$x = 0$

Critical point values:

Putting in equation (i)

$f(-1) = (-1)^{4/3} = \left[(-1)^4\right]^{1/3} = (1)^{1/3} = 1$

End point values:

Putting in equation (i)

$f(8) = (8)^{4/3} = \left[(2)^3\right]^{4/3} = (2)^4 = 16$

$f(0) = 0$

From the values: $1, 16, 0$

- The Global Maximum is $f(8) = 16$
- The Global Minimum is $f(0) = 0$

4. Find the global and local extrema of the following functions over the specified domains. (Hint: Use the increasing and decreasing behavior of the functions including the end points).

(i) $f(x) = 4 - x^2$ on $[-1, 2]$ **12302086**

Solution:

$f(x) = 4 - x^2$ on $[-1, 2]$

$f(x) = 4 - x^2$ .....(i)

Differentiate w.r.t. 'x',

$f'(x) = \frac{d}{dx}\left(4 - x^2\right)$

$f'(x) = -2x$

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Put $f'(x)=0$

$-2x=0$

$x=0$

$f(-1)=4-(-1)^2=4-1=3$

$f(2)=4-(2)^2=4-4=0$

$f(0)=4-(0)^2=4$

and $f''(x)=-2<0$

Local Maximum is 4 at $x=0$

Global Maximum is 4 at $x=0$

Global Minimum is 0 at $x=2$

(ii) $f(x)=\cos x-\sin x$ on $[0,2\pi]$

Solution: 12302087

$f(x)=\cos x-\sin x$ on $[0,2\pi]$

$f'(x)=-\sin x-\cos x$

$f'(x)=0$

$-\sin x-\cos x=0$

$\cos x+\sin x=0$

$\sin x=-\cos x$

$\tan x=-1$

$x=\frac{3\pi}{4},\frac{7\pi}{4}$

Now $f(0)=\cos(0)-\sin(0)$

$f(0)=1$

$f(2\pi)=\cos 2\pi-\sin 2\pi$

$f(2\pi)=1$

$f\left(\frac{3\pi}{4}\right)=\cos\frac{3\pi}{4}-\sin\frac{3\pi}{4}$

$=-\sqrt{2}$

$f\left(\frac{7\pi}{4}\right)=\cos\frac{7\pi}{4}-\sin\frac{7\pi}{4}$

$=\sqrt{2}$

Now

$f''(x)=-\cos x-(-\sin x)$

$=-\cos x+\sin x$

$f''\left(\frac{3\pi}{4}\right)=-\cos\frac{3\pi}{4}+\sin\frac{3\pi}{4}$

$=-\left[\cos\left(\pi-\frac{\pi}{4}\right)+\sin\left(\pi-\frac{\pi}{4}\right)\right]$

$=\left(-\cos\frac{\pi}{4}\right)+\sin\frac{\pi}{4}$

$=\cos\frac{\pi}{4}+\sin\frac{\pi}{4}$

$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}>0$

$f''\left(\frac{7\pi}{4}\right)=-\cos\frac{7\pi}{4}+\sin\frac{7\pi}{4}$

$=-\left[\cos\left(2\pi-\frac{\pi}{4}\right)\right]+\left[\sin\left(2\pi-\frac{\pi}{4}\right)\right]$

$=-\left(\cos\frac{\pi}{4}\right)-\left(\sin\frac{\pi}{4}\right)$

$=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=-\sqrt{2}<0$

Local Minimum at $x=\frac{3\pi}{4}$ is $\sqrt{2}$

Local Maximum at $x=\frac{7\pi}{4}$ is $\sqrt{2}$

Global Maximum at $x=\frac{7\pi}{4}$ is $\sqrt{2}$

Global Minimum at $x=\frac{3\pi}{4}$ is $-\sqrt{2}$

5. Prove that $f(x)=\frac{\ln(2x)}{x^2}$ has local maximum at $x=\frac{\sqrt{e}}{2}$. 12302088

Solution:

$f(x)=\frac{\ln(2x)}{x^2}$ has local maximum at $x=\frac{\sqrt{e}}{2}$

$f(x)=\frac{\ln(2x)}{x^2}$

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Differentiate w.r.t. 'x':

$$f'(x)=\frac{d}{dx}\left[\frac{\ln(2x)}{x^2}\right]$$

$$f'(x)=\frac{x^2\frac{d}{dx}\ln(2x)-\ln(2x)\frac{d}{dx}(x^2)}{(x^2)^2}$$

$$\because \left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}$$

$$f'(x)=\frac{x^2\left(\frac{1}{2x}\cdot 2\right)-\ln(2x)(2x)}{(x)^4}$$

$$f'(x)=\frac{x-2x\ln 2x}{x^4}$$

$$f'(x)=x\left(\frac{1-2\ln 2x}{x^4}\right)$$

$$f'(x)=\frac{1-2\ln 2x}{x^3}\ldots\ldots\ldots(i)$$

For stationary points:

Put $f'(x)=0$

$$\frac{x(1-2\ln x)}{x^3}=0$$

$$1-2\ln(2x)=0$$

$$-2\ln(2x)=-1$$

$$2\ln(2x)=1$$

$$\ln(2x)=\frac{1}{2}$$

Taking exponential on both sides:

$$2x=e^{1/2}$$

$$x=\frac{1}{2}e^{1/2}$$

$$x=\frac{\sqrt{e}}{2}$$

Now, again differentiate (i) w.r.t. 'x':

$$f''(x)=\frac{d}{dx}\left(\frac{1-2\ln 2x}{x^3}\right)$$

$$f''(x)=\frac{d}{dx}(1-2\ln 2x)x^{-3}$$

$$f''(x)=(1-2\ln 2x)\frac{d}{dx}(x^{-3})+x^{-3}\frac{d}{dx}(1-2\ln 2x)$$

$$f''(x)=(1-2\ln 2x)(-3x^{-4})+x^{-3}\left(0-2\cdot\frac{1}{2x}\cdot 2\right)$$

$$=x^{-4}(-3+6\ln 2x)-2x^{-4}$$

$$f''(x)=\frac{6\ln 2x-5}{x^4}$$

$$\because x=\frac{\sqrt{e}}{2}$$

$$2x=\sqrt{e}$$

Taking 'ln' both sides:

$$\ln 2x=\ln(e)^{1/2}$$

$$\ln 2x=\frac{1}{2}\ln e$$

$$\ln 2x=\frac{1}{2}$$

At $x=\frac{\sqrt{e}}{2}$

$$f''\left(\frac{\sqrt{e}}{2}\right)=\frac{6\left(\frac{1}{2}\right)-5}{\left(\frac{\sqrt{e}}{2}\right)^4}$$

$$f''\left(\frac{\sqrt{e}}{2}\right)=\frac{3-5}{\left(\frac{\sqrt{e}}{2}\right)^4}$$

$$f''\left(\frac{\sqrt{e}}{2}\right)=\frac{-2}{\left(\frac{\sqrt{e}}{2}\right)^4}<0$$

Hence $f(x)$ has local maximum at $x=\frac{\sqrt{e}}{2}$.

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6. Show that $f(x)=x^{x^2}$ has a relative minimum at $x=\frac{1}{\sqrt{e}}$. **12302089**

Solution:

$f(x)=x^{x^2}$ at $x=\frac{1}{\sqrt{e}}$

Taking 'ln' on both sides.

$\ln f(x)=\ln\left(x^{x^2}\right)$

$\ln f(x)=x^2\ln x$

Differentiate w.r.t. to 'x':

$\frac{d}{dx}\left[\ln f(x)\right]=\frac{d}{dx}\left(x^2\ln x\right)$

$\frac{1}{f(x)}f'(x)=x^2\left(\frac{1}{x}\right)+\ln(x)(2x)$

$f'(x)=f(x)\left[x+2x(\ln x)\right]$

$f'(x)=\left(x^{x^2}\right)(x+2x\ln x)$ .....(i)

Again Differentiate w.r.t. 'x':

$\frac{d}{dx}\left[f'(x)\right]=\frac{d}{dx}\left[x^{x^2}(x+2x\ln x)\right]$

$f''(x)=\frac{d}{dx}\left[f(x)(x+2x\ln x)\right]$

$=f(x)\cdot\frac{d}{dx}(x+2x\ln x)+(x+2x\ln x)\frac{d}{dx}\left[f(x)\right]$

$=f(x)\left[1+2\left(x\cdot\frac{1}{x}+\ln x\right)\right]+(x+2x\ln x)f'(x)$

$=f(x)(1+2+2\ln x)+(x+2x\ln x)(x+2x\ln x)x^{x^2}$

Using equation (i), at $x=\frac{1}{\sqrt{e}}$

$=f(x)(3+2\ln x)+f(x)(x+2x+\ln x)^2$

$=f(x)\left[(3+2\ln x)+(x+2x\ln x)^2\right]$

$=f(x)\left[\left(3+2\ln\cdot\frac{1}{\sqrt{e}}\right)+\left(\frac{1}{\sqrt{e}}+2\cdot\frac{1}{\sqrt{e}}\ln\frac{1}{\sqrt{2}}\right)^2\right]$

$=f(x)\left[3+2\left(\frac{-1}{2}\right)+\left(\frac{1}{\sqrt{e}}+2\cdot\frac{1}{\sqrt{e}}\left(\frac{-1}{2}\right)\right)^2\right]$

$=f(x)=(3-1)+(0)^2$

$=f(x)(2)$

$=2f(x)>0$

Thus $f(x)$ has relative minima at $x=\frac{1}{\sqrt{e}}$.

So $f(x)$ is relative minimum at $x=\frac{1}{\sqrt{e}}$.

Linearization

If a function $f(x)$ is differentiable at $x=a$ then the approximating function

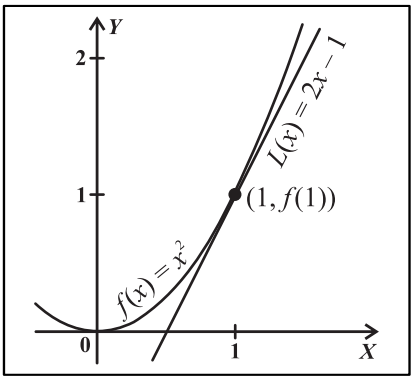
$$L(x)=f(a)+f'(a)(x-a)$$

is the linearization of $f(x)$ at $x=a$. The approximation

$$f(x)\approx L(x)$$

of $f(x)$ by $L(x)$ is called the local linear approximation of $f(x)$ at $x=a$. The point $x=a$ is referred as the center of the approximation.

We can see in figure the tangent line $L(x)=2x-1$ to the graph of the curve $f(x)=x^2$ at $(1,f(1))$ lies close to the curve near the point of tangency. For a small interval on either side of $x=1$, y-values of the tangent line provide good approximations to the y-values of the curve $f(x)=x^2$. We observe this phenomenon by observing the table of values for the difference between the values of $L(x)$ and $f(x)$. The phenomenon is true not just for this curve; every differentiable curve behaves locally like its tangent line.



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x	0.7	0.8	0.9	1	1.1	1.2	1.3
$f(x)=x^2$	0.49	0.64	0.81	1	1.21	1.44	1.69
$L(x)=2x-1$	0.4	0.6	0.8	1	1.2	1.4	1.6
$f(x)-L(x)$	0.09	0.04	0.01	0	0.01	0.04	0.09

Example 22: (a) Find the local linear approximation of $f(x)=\sqrt{2x-1}$ at $x=5$.
(b) Use the local linear approximation obtained in part (a) to approximate $\sqrt{9.2}$, and compare your approximation to the result produced directly by calculator.

Solution: 12302090

(a) We are given:

$f(x)=\sqrt{2x-1}$ .....(i)

Differentiate $f(x)$ w.r.t. x:

$$f'(x)=\frac{d}{dx}\left(\sqrt{2x-1}\right)$$

$$f'(x)=\frac{d}{dx}(2x-1)^{1/2}$$

$$f'(x)=\frac{1}{2}(2x-1)^{\frac{1}{2}}(2)$$

$$f'(x)=\frac{1}{\sqrt{2x-1}} \quad \text{.....(ii)}$$

Substitute $x=5$ into equations (i) and (ii):

$$f(5)=\sqrt{2(5)-1}$$
$$=\sqrt{10-1}=\sqrt{9}$$
$$=3$$

And

$$f'(5)=\frac{1}{\sqrt{2(5)-1}}$$
$$=\frac{1}{\sqrt{10-1}}=\frac{1}{\sqrt{9}}$$
$$=\frac{1}{3}$$

Now linearization is:

$$L(x)=f(5)+f'(5)(x-5)$$

$$\boxed{\because L(x)=f(a)+f'(a)(x-a)}$$

$$L(x)=3+\frac{x}{3}-\frac{5}{3}$$
$$=\frac{9+x-5}{3}$$
$$=\frac{4+x}{3}$$
$$L(x)=\frac{1}{3}(x+4)$$

Thus, the local linear approximation at $x=5$ is:

$$f(x)\approx L(x)$$

$$\sqrt{2x-1}\approx\frac{1}{3}(x+4) \quad \text{.....(iii)}$$

(b) Substitute $x=5.1$ into equation (iii)

$$\sqrt{2(5.1)-1}\approx\frac{1}{3}(5.1+4)$$

$$\sqrt{9.2}\approx\frac{1}{3}(9.1)$$

$$\sqrt{9.2}\approx 3.0333 \text{ .....(iv)}$$

Calculator value $\sqrt{9.2}=3.0331$ .....(v)

From expressions (iv) and (v), we see that the calculator value is very close to the value obtained by linear approximation.

Example 23: (a) Find the local linear approximation of $f(x)=\cos x$ at $x=\frac{\pi}{2}$.
(b) Use the local linear approximation obtained in part (a) to approximate $\cos 91°$, and compare your approximation to the result produced directly by a calculator. 12302091

Solution:

(a) We are given:

$f(x)=\cos x$ .....(i)

Differentiate with respect to x:

$f'(x)=-\sin x$ .....(ii)

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Substitute $x=\frac{\pi}{2}$ into the equations (i) and (ii)

$f\left(\frac{\pi}{2}\right)=\cos\frac{\pi}{2}=0$ and

$f'\left(\frac{\pi}{2}\right)=-\sin\frac{\pi}{2}=-1$

The linearization at $x=\frac{\pi}{2}$ is:

$$L(x)=f\left(\frac{\pi}{2}\right)+f'\left(\frac{\pi}{2}\right)\left(x-\frac{\pi}{2}\right)$$

$$\boxed{\because L(x)=f(a)+f'(a)(a-x)}$$

$$=0+(-1)\left(x-\frac{\pi}{2}\right)$$

$$\Rightarrow L(x)=-x+\frac{\pi}{2} \quad \ldots\ldots\ldots(iii)$$

Thus, the local linear approximation at $x=\frac{\pi}{2}$ is:

$$f(x)\approx L(x)$$

From equations (i) and (iii):

$\cos x\approx -x+\frac{\pi}{2}$ .....(iv)

(b) To approximate $\cos 91°$, first convert degree into radians:

$$91°=91\times\frac{\pi}{180}=\frac{91\pi}{180}$$

Substitute $x=91°=1.588825$ into equation (iv):

$\cos 91°\approx\frac{-91\pi}{180}+\frac{\pi}{2}$

$\Rightarrow \cos 91°\approx -0.01745329$ .....(v)

Calculator value:

$\cos 91°=-0.01745240$ .....(vi)

From expression (iv) and (v), we see that the calculator value is very close to the value obtained by linear approximation.

EXERCISE 2.5

1. Find the linear approximation of $f(x)=\sqrt{x+1}$ at $x=3$, and use it to approximate $\sqrt{3.1}$ and $\sqrt{2.9}$. 12302092

Solution:

$f(x)=\sqrt{x+1}$ .....(i)

Differentiate w.r.t. 'x':

$f'(x)=\frac{1}{2}(x+1)^{-1/2}$

$f'(x)=\frac{1}{2\sqrt{x+1}}$ .....(ii)

Substitute $x=3$ in equation (i) and (ii)

$f(3)=\sqrt{3+1}=\sqrt{4}=2$

$f'(3)=\frac{1}{2\sqrt{3+1}}=\frac{1}{2\sqrt{4}}=\frac{1}{4}$

The Linearization at $x=3$

$L(x)=f(3)+f'(3)(x-3)$

$\because L(x)=f(a)+f'(a)(x-3)$

$L(x)=2+\frac{1}{4}(x-3)$

$L(x)=\frac{8+x-3}{4}$

$L(x)=\frac{x+5}{4}$

The local linear approximation at $x=2.1$ is

$f(x)\approx L(x)$

$\sqrt{x+1}\approx\frac{x+5}{4}$

$\sqrt{2.1+1}\approx\frac{2.1+5}{4}$

$\sqrt{3.1}\approx\frac{7.1}{4}$

$\sqrt{3.1}\approx 1.775$

The local linear approximation at $x=1.9$ is

$\sqrt{x+1}\approx\frac{x+5}{4}$

$\sqrt{1.9+1}\approx\frac{1.9+5}{4}$

$\sqrt{2.9}\approx 1.725$

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2. (a) Find the local linear approximation of $f(x)=\sin x$ at $x=0$. 12302093

Solution:

$f(x)=\sin x$ .....(i)

Differentiate w.r.t. 'x':

$f'(x)=\cos x$ .....(ii)

Substitute $x=0$ in equation (i) and (ii)

$f(0)=\sin(0)=\sin 0=0$

$f'(0)=\cos(0)=\cos 0=1$

The linearization at $x=0$

$L(x)=f(0)+f'(0)(x-0)$

$\because L(x)=f(a)+f'(a)(x-a)$

$=0+1(x-0)$

$L(x)=x$ .....(iii)

(b) Use the local linear approximation obtained in part (a) to approximate $\sin 2°$, and compare your approximation to the result produced directly by a calculator. 12302094

Solution:

To approximate $\sin 2°$ first convert it into radian.

$2°=2\times\frac{\pi}{180}rad.$ $\quad\because 1°=\frac{\pi}{180}rad.$

$2°=\frac{\pi}{90}\approx 0.0349$

The local linearization:

$f(x)\approx L(x)$

$\sin 2°\approx x$

Fusing (iii)

Substitute $x=2°=0.0349$

$\sin 2°\approx x$

$\sin 2°=0.0349$ .....(iv)

Using calculator:

$\sin 2°\approx 0.0348$ .....(v)

From approximation (iv) and (v) we see that the calculator values is very close to the value obtained by the linear approximation.

3. Find the linear approximation of the following functions at the given points:

(i) $f(x)=\sqrt{x^2+16}$ at $x=3$ 12302095

Solution:

$f(x)=\sqrt{x^2+16}$ at $x=3$

$f(x)=\sqrt{x^2+16}$ ..... (i)

Differentiate w.r.t. 'x':

$\frac{d}{dx}(f(x))=\frac{d}{dx}\sqrt{x^2+16}$

$f'(x)=\frac{1}{2}(x^2+16)^{\frac{1}{2}-1}\frac{d}{dx}(x^2+16)$

$f'(x)=\frac{1}{2}(x^2+16)^{\frac{-1}{2}}(2x+0)$

$f'(x)=\frac{x}{\sqrt{x^2+16}}$ ..... (ii)

Substitute $x=3$ in equation (i) and (ii)

$f(3)=\sqrt{3^2+16}=\sqrt{9+16}=\sqrt{25}=5$

$f'(x)=\frac{3}{\sqrt{3^2+16}}=\frac{3}{\sqrt{9+16}}=\frac{3}{25}=\frac{3}{5}$

The linearization at $x=3$

$L(x)=f(3)+f'(3)(x-3)$

$=5+\frac{3}{5}(x-3)$

$=\frac{25+3x-9}{5}$

$L(x)=\frac{3x+16}{5}$

(ii) $f(x)=\tan x$ at $x=\pi$ 12302096

Solution:

$f(x)=\tan x$ at $x=\pi$

$f(x)=\tan x$ .....(i)

Differentiate w.r.t 'x':

$\frac{d}{dx}[f(x)]=\frac{d}{dx}(\tan x)$

$f'(x)=\sec^2 x$ .....(ii)

Substitute $x=\pi$ in equation (i) and (ii)

$f(\pi)=\tan\pi=0$

$f'(x)=\sec^2\pi=1$

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The linearization at $x=\pi$

$L(x)=f(\pi)+f'(\pi)(x-\pi)$

$L(x)=0+(1)(x-\pi)$

$L(x)=x-\pi$

(iii) $f(x)=\cos x$ **at** $x=0$ **12302097**

Solution:

$f(x)=\cos x$ at $x=0$

$f(x)=\cos x$ .....(i)

Differentiate w.r.t. 'x':

$\frac{d}{dx}[f(x)]=\frac{d}{dx}(\cos x)$

$f'(x)=-\sin x$ .....(ii)

Substitute $x=0$ in (i) and (ii)

$f(0)=\cos(0)=1$

$f'(0)=-\sin 0=0$

The linearization at $x=0$

$L(x)=f(0)+f'(0)(x-0)$

$L(x)=1+0(x-0)$

$L(x)=1$

4. Use a local linear approximation to estimate the following:

(i) $(82)^{\frac{1}{3}}$ **12302098**

Solution:

$(82)^{\frac{1}{3}}$ .....(i)

Let $f(x)=(x)^{\frac{1}{3}}$ where $a=64,\ x=82$

Differentiate w.r.t. 'x':

$\frac{d}{dx}\left[f(x)=\frac{d}{dx}\left(x^{1/3}\right)\right]$

$f'(x)=\frac{1}{3}x^{\frac{1}{3}-1}$

$f'(x)=\frac{1}{3}x^{\frac{-2}{3}}$

$f'(x)=\frac{1}{3x^{\frac{2}{3}}}$ .....(ii)

Substitute $x=64$ in (i) and (ii)

$f(64)=(64)^{1/3}=\left(4^3\right)^{\frac{1}{3}}=4$

$f'(64)=\frac{1}{3(64)^{\frac{2}{3}}}=\frac{1}{3\left(4^3\right)^{\frac{2}{3}}}$

$=\frac{1}{3(4)^2}=\frac{1}{3(16)}$

$=\frac{1}{48}$

The linearization at $x=64$

$L(x)=f(64)+f'(64)(x-64)$

$=4+\frac{1}{48}(x-64)$

$L(x)=\frac{192+x-64}{48}$

$L(x)=\frac{x+128}{48}$

The local linear approximation:

$\because f(x)\approx L(x)$

$x^{1/3}\approx\frac{x+128}{48}$

at $x=82$

$(82)^{1/3}\approx\frac{82+128}{48}$

$(82)^{1/3}\approx\frac{210}{48}$

$(82)^{1/3}\approx 4.375$

(ii) $\sin 29°$ **12302099**

Solution:

$\sin 29°$

Let $f(x)=\sin x$ .....(i)

Differentiate w.r.t. 'x':

$f'(x)=\cos x$ .....(ii)

where $a=30°$ *or* $\frac{\pi}{6}$

Substitute $a=\frac{\pi}{6}$ in (i) and (ii)

$f\left(\frac{\pi}{6}\right)=\sin\frac{\pi}{6}=\frac{1}{2}$

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$$f'\left(\frac{\pi}{6}\right)=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$$

The linearization at $x=\frac{\pi}{6}$

$$L(x)=f\left(\frac{\pi}{6}\right)+f'\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right)$$

$$L(x)=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)$$

$x=29°$ First we convert it into radian

$$29°=\frac{29\pi}{180}=\frac{29}{180}\pi Rad.$$

The local linearization approximation is at

$$x=\frac{29\pi}{180}Rad. \qquad \boxed{\because f(x)=L(x)}$$

$$f(x)\approx x$$

$$\sin 29°\approx\frac{1}{2}+\frac{\sqrt{3}}{2}\left(\frac{29\pi}{180}-\frac{\pi}{6}\right)$$

$$\approx\frac{1}{2}+\frac{\sqrt{3}}{2}\left(\frac{29\pi-30\pi}{180}\right)$$

$$\approx\frac{1}{2}+\frac{\sqrt{3}}{2}\left(\frac{-\pi}{180}\right)$$

$$\sin 29°\approx 0.485$$

Differentials

Assume that f is differentiable at a point x, if dx is an independent variable that can take values in the neighborhood of 0 and not 0, then we define dependent variable dy, by the formula:

$$dy=f'(x)dx \quad \ldots\ldots\ldots(i)$$

Since $dx\neq 0$, we can divide both sides of (i) by dx to obtain:

$$\frac{dy}{dx}=f'(x)$$

Example 24: **(a) Find dy if $y=x^4+7x$.**
(b) Find the value of dy when $x=1$ and $dx=0.1$. **12302100**

Solution: (a) Let $f(x)=x^4+7x$.

Differentiate $f(x)$ with respect to x:

$$f'(x)=4x^3+7 \ldots\ldots\ldots(i)$$

We know: $dy=f'(x)dx$

Substitute the value of $f'(x)$ from equation (i):

$$dy=\left(4x^3+7\right)dx \ldots\ldots\ldots(ii)$$

(b) Substitute $x=1$ and $dx=0.1$ into equation (ii):

$$dy=\left(4(1)^3+7\right)(0.1)=11(0.1)=1.1$$

Thus, the value of dy is 1.1.

Using dy to Approximate Δy

When a function $y=f(x)$ is differentiable at a point $P(x,y)$, we can use the differential dy to estimate the actual change Δy in y, for small change Δx in x.

The actual change Δy in y is given by:

$$\Delta y=f(x+\Delta x)-f(x)\ldots\ldots\ldots(i)$$

But calculating Δy exactly often requires complicated formulas. Instead, we use the differential dy, defined as:

$$dy=f'(x)dx$$

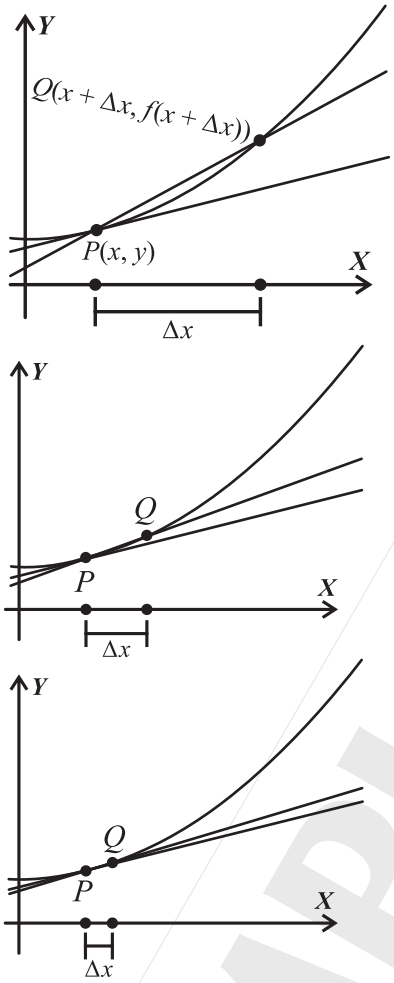
If we take $dx=\Delta x$, then: $dy=f'(x)\Delta x$ …(ii)

Let $Q\left(x+\Delta x, f(x+\Delta x)\right)$ be a neighboring point of $P(x,y)$ on $y=f(x)$.

- The slope of the tangent line at P is $f'(x)$
- The slope of the secant line connecting P and Q is $\frac{\Delta y}{\Delta x}$

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As $\Delta x \to 0$, the point Q approaches P, and secant line PQ approaches the tangent line at P.



Therefore, for small Δx:

Slope of tangent line at $P \approx$ Slope of secant line PQ

$$\Rightarrow f'(x) \approx \frac{\Delta y}{\Delta x}$$

Substitute $f'(x) \approx \frac{\Delta y}{\Delta x}$ into equation (ii):

$$dy \approx \frac{\Delta y}{\Delta x} \times \Delta x$$

$$dy \approx \Delta y \text{ (for } \Delta x \text{ is small)}$$

Thus, for small change Δx the actual change Δy is approximately equal to the differential dy, that is:

$$\Delta y \approx dy = f'(x)dx$$

Error in Differential Approximations

Suppose f is a differentiable function at a point x and there is a small change Δx in x if:

- Δy is actual corresponding change in y, given by
$$\Delta y = f(x+\Delta x) - f(x)$$
- dy is the approximation change in y obtained using differentials, given by
$$dy = f'(x)dx \text{ where } dx = \Delta x$$

Then:

Error in Δy is: Error $= \Delta y - dy$

Relative error in Δy is: Relative Error $= \frac{\Delta y - dy}{\Delta y} = \frac{Error}{Original\,Value}$, when $\Delta y \neq 0$

Percentage error Δy is:

$$\text{Percentage Error} = \left|\frac{\Delta y - dy}{\Delta y}\right| \times 100$$

$$= \left|\frac{Error}{Original\,Value}\right| \times 100 \text{ when } \Delta y \neq 0$$

Example 25: Find dy and Δy for the function $f(x) = x^3$ when $x = 1$ and $dx = 0.1$. **12302101**

Solution:

We are given:

$$f(x) = x^3 \ldots\ldots\ldots\text{(i)}$$

Differentiate with respect to x:

$$f'(x) = 3x^2$$

The formula for dy is:

$$dy = f'(x)dx$$

Substituting expression for $f'(x)$:

$$dy = 3x^2 dx$$

Substituting $x = 1$ and $dx = 0.1$:

$$dy = 3(1)^2(0.1) = 0.3$$

We know:

$$\Delta y = f(x+\Delta x) - f(x)$$

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Substitute $x=1$ and $dx=0.1$:

$\Delta y = f(1+0.1)-f(1) = f(1.1)-f(1)$

From equation (i), $f(x)=x^3$, so:

$\Delta y = (1.1)^3-(1)^3 = 1.331-1 = 0.331$

Example 26: Use differentials to approximate $\sqrt{24}$. 12302102

Solution:

Let $f(x)=\sqrt{x}$ .....(i)

Differentiate with respect to x:

$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$ .....(ii)

Choose a real number that closest to 24 is a perfect square. Take $x=25$

We are approximating $\sqrt{24}$, so:

$x+\Delta x=24$ and $x=25$

Thus: $24+\Delta x=25 \Rightarrow \Delta x=-1$

Since in differential approximation, $dx=\Delta x$, we take: $dx=-1$

We know: $dy=f'(x)dx$

From equation (ii):

$$dy=\frac{1}{2\sqrt{x}}dx$$

Substitute $x=25$ and $dx=-1$:

$$dy=\frac{1}{2\sqrt{25}}(-1)=-\frac{1}{10}=-0.1$$

We know: $f(x+\Delta x)\approx f(x)+dy$

Substitute $dy=-0.1, x=25$ and $\Delta x=-1$:

$f(25-1)\approx f(25)+(-0.1)$

$f(24)\approx f(25)-0.1$

From equation (i):

$\sqrt{24}=\sqrt{25}-0.1=5-0.1=4.9$

Example 27: Using differentials approximate the value of $\cos 61°$. 12302103

Solution:

Let $f(x)=\cos x$ .....(i)

Differentiate with respect to x:

$f'(x)=-\sin x$ .....(ii)

Choose $x=60°$ because calculating $\cos 60°$ is easy.

We are approximating $\cos 61°$, so:

$x+\Delta x=61°$ and $x=60°$

Thus: $60°+\Delta x=61° \Rightarrow \Delta x=1°$

Since in differential approximation, $dx=\Delta x$,

we take: $dx=1°=\frac{\pi}{180}$ radian

We know: $dy=f'(x)dx$

From equation (ii):

$$dy=-\sin x dx$$

Substitute $x=60°$ and $dx=\frac{\pi}{180}$:

$$dy=-\sin 60°\left(\frac{\pi}{180}\right)=-\frac{\sqrt{3}}{2}\times\frac{\pi}{180}=-0.0151$$

We know: $f(x+\Delta x)\approx f(x)+dy$

Substitute $dy=-0.0151, x=60°$ and $x+\Delta x=61°$: $f(61°)\approx f(60°)+(-0.0151)$

From equation (i):

$\cos 61° = \cos 60° - 0.0151 = 0.5 - 0.0151 = 0.4849$

Example 28: The side of a cube is measured to be 25 cm, with a possible error of ± 1 cm.

(a) Use differentials to estimate the error in the calculated volume.

(b) Estimate the percentage errors in the side and volume.

Solution: 12302104

Let x be side of the cube:

$x=25cm,\ dx=\Delta x=\pm 1cm$

(a) The differential of V is:

$$dV=\frac{d}{dx}(x^3)dx$$

$$dV=3x^2dx$$ .....(i)

Substitute $x=25$ and $dx=\pm 1$ into equation (i):

$dV=3(25)^2(\pm 1)=\pm 1875cm^3$

(b) Percentage error in measuring the side is:

$\left|\frac{dx}{x}\right|\times 100=\frac{1}{25}\times 100=4\%$ .....(ii)

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Since: $V = x^3$ and $dV = 3x^2 dx$

The percentage error in measuring the volume is:

$$\left|\frac{dV}{V}\right| \times 100 = \left|\frac{3x^2 dx}{x^3}\right| \times 100 = 3 \times \left|\frac{dx}{x}\right| \times 100$$

Using equation (ii):

$$\left|\frac{dV}{V}\right| \times 100 = 3 \times 4\% = 12\%$$

Example 29: The radius *r* of a circle is measured with an error of at most 2%. What is the maximum corresponding percentage error in computing the circles. (a) circumference? (b) area? 12302105

Solution:

Since *r* is the radius. The percentage error in measuring the radius is given as:

$$\left|\frac{dr}{r}\right| \times 100 = 2\% \ldots\ldots\ldots\text{(i)}$$

(a) Let *C* be the circumference of the circle:

$$C = 2\pi r$$

The differential of *C* is:

$$dC = \frac{d}{dr}(2\pi r)dr = 2\pi dr \ldots\ldots\ldots\text{(ii)}$$

Percentage error in measuring the circumference:

$$\left|\frac{dC}{C}\right| \times 100 = \left|\frac{2\pi dr}{2\pi r}\right| \times 100 = \left|\frac{dr}{r}\right| \times 100$$

Using equation (i):

$$\left|\frac{dC}{C}\right| \times 100 = 2\%$$

(b) Let *A* be the area of the circle: $A = \pi r^2$

The differential of A is:

$$dA = \frac{d}{dr}(\pi r^2)dr = 2\pi r dr$$

Percentage error in measuring the area:

$$\left|\frac{dA}{A}\right| \times 100 = \left|\frac{2\pi r dr}{\pi r^2}\right| = 2 \times \left|\frac{dr}{r}\right| \times 100$$

Using equation (i):

$$\left|\frac{dA}{A}\right| \times 100 = 2 \times 2\% = 4\%$$

EXERCISE 2.6

1. Find the differential *dy* in the following cases:

(i) $y = x^3 - 3\sqrt{x}$ 12302106

Solution:

$y = x^3 - 3\sqrt{x}$

$\boxed{\because y = f(x)}$

$f(x) = x^3 - 3\sqrt{x}$

Differentiate w.r.t. '*x*':

$$f'(x) = \frac{d}{dx}\left(x^3 - 3\sqrt{x}\right)$$

$$= \frac{d}{dx}\left(x^3\right) - 3\frac{d}{dx}\left(\sqrt{x}\right)$$

$$= 3x^2 - 3\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$$

$$= 3x^2 - \frac{3}{2}x^{\frac{-1}{2}}$$

$$f'(x) = 3x^2 - \frac{3}{2\sqrt{x}} \ldots\ldots\ldots\text{(i)}$$

The formula for dy

$dy = f'(x)dx$

$$\boxed{dy = \left(3x^2 - \frac{3}{2\sqrt{x}}\right)dx}$$

(ii) $y = \cos x^2$ 12302107

Solution:

$y = \cos x^2$

$f(x) = \cos^2 x$ $\boxed{\because y = f(x)}$

Differentiate w.r.t. '*x*':

$$f'(x) = \frac{d}{dx}\cos^2 x$$

$$f'(x) = -\sin x^2 \frac{d}{dx}\left(x^2\right)$$

$$f'(x) = -2x\sin x^2 \ldots\ldots\ldots\text{(i)}$$

The formula for dy is:

$dy = f'(x)dx$

$$\boxed{dy = -2x\sin\left(x^2\right)dx}$$

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2. Use dy to approximate Δy for the following functions:

(i) $y = x^3 - x$ when x changes from 3 to 3.02. **12302108**

Solution:

$y = x^3 - x$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^3 - x)$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) - \frac{d}{dx}(x)$$

$$\frac{dy}{dx} = (3x^2 - 1)$$

$$dy = (3x^2 - 1)dx \text{ where}$$

$x = 3,\ dx = 3.02 - 3 = 0.02$

$$dy = \left[3(3)^2 - 1\right](0.02)$$

$$dy = \left[3(9) - 1\right](0.02) = (26)(0.02)$$

$$dy = 0.52$$

$$\boxed{\Delta y \approx dy = 0.52}$$

(ii) $y = \sin x + \cos x$ when x changes from $30°$ to $29°$. **12302109**

Solution:

$y = \sin x + \cos x$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y) = \frac{d}{dx}(\sin x + \cos x)$$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x) + \frac{d}{dx}(\cos x)$$

$$\frac{dy}{dx} = \cos x - \sin x$$

$$dy = (\cos x - \sin x)dx$$

where

$$x = 30°, dx = 29° - 30° = -1° = \frac{-\pi}{180} radians$$

$$dy = (\cos 30° - \sin 30°)\left(\frac{-\pi}{180}\right)$$

$$= \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)\left(\frac{-\pi}{180}\right)$$

$$= \left(\frac{\sqrt{3} - 1}{2}\right)\left(\frac{-\pi}{180}\right)$$

$$= \left(\frac{1.73 - 1}{-2}\right)\left(\frac{-\pi}{180}\right)$$

$$= \frac{-0.73\pi}{360}$$

$$= -0.0064$$

$$\Delta y \approx dy = -0.0064$$

(iii) $y = \frac{1}{x}$ when $x = 0.5$ and $dx = -0.1$

Solution: **12302110**

$$y = \frac{1}{x}$$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-1})$$

$$\frac{dy}{dx} = x^{-2}$$

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

$$dy = -\frac{1}{x^2}dx$$

where $x = 0.5,\ dx = -0.1$

$$dy = -\frac{1}{(0.5)^2}(-0.1)$$

$$dy = \frac{-1}{0.25}(-0.1)$$

$$dy = (-4)(-0.1)$$

$$dy = 0.4$$

$$\Delta y \approx dy = 0.4$$

3. Use differentials to approximate the value of:

(i) $\sqrt{26}$ **12302111**

Solution:

Let $f(x) = \sqrt{x}$ .....(i)

Differentiate w.r.t. 'x':

$$f'(x) = \frac{d}{dx}(\sqrt{x})$$

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$$f'(x)=\frac{d}{dx}(x)^{\frac{1}{2}}$$
$$f'(x)=\frac{1}{2}x^{\frac{1}{2}-1}$$
$$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}$$
$$f'(x)=\frac{1}{2\sqrt{x}} \quad \ldots\ldots\ldots(ii)$$

Choose a number closest to 26 which is perfect square, take $x=25$

We are approximating $\sqrt{26}$, so:

$x+\Delta x=26$
$25+\Delta x=26$
$\Delta x=26-25$
$\Delta x=1$

Since in differential approximation
$\Delta x=dx$, So, $dx=1$

We know that:
$dy=f'(x)dx$

Putting values
$$dy=\frac{1}{2\sqrt{x}}(1)$$
$$dy=\frac{1}{2\sqrt{25}}=\frac{1}{2(5)}=\frac{1}{10}$$
$$dy=0.1$$

We know that
$f(x+\Delta x)\approx f(x)+dy$
$f(25+1)\approx f(25)+0.1$
$f(26)\approx f(25)+0.1$

From equation (i),
$\sqrt{26}\approx\sqrt{25}+0.1=5+0.1=5.1$

(ii) $(33)^{\frac{1}{5}}$ **12302112**

Solution:

$(33)^{\frac{1}{5}}$

Let $f(x)=(x)^{\frac{1}{5}}$ .....(i)

Differentiate w.r.t. 'x':

$$f'(x)=\frac{d}{dx}\left[(x)^{1/5}\right]$$
$$f'(x)=\frac{1}{5}x^{\frac{1}{5}-1}$$
$$f'(x)=\frac{1}{5}x^{\frac{-4}{5}}$$
$$f'(x)=\frac{1}{5x^{4/5}} \quad \ldots\ldots\ldots(ii)$$

Choose a number closest to 33 which has perfect fifth root. Take $x=32$

$x+\Delta x=33$
$32+\Delta x=33$
$\Delta x=33-32$
$\boxed{\Delta x=1}$

Since in differential approximation $\Delta x=dx$
So, $\boxed{dx=1}$

We know that
$dy=f'(x)dx$

Using (ii),
$$dy=\frac{1}{5(x)^{\frac{4}{5}}}dx$$

Substitute $x=32, dx=1$
$$dy=\frac{1}{5(32)^{\frac{4}{5}}}(1)=\frac{1}{5\left(2^5\right)^{\frac{4}{5}}}$$
$$=\frac{1}{5(2)^4}=\frac{1}{5(16)}$$
$$dy=\frac{1}{80}$$

We know that
$f(x+\Delta x)\approx f(x)+dy$
$f(32+1)\approx f(32)+\frac{1}{80}$
$f(33)\approx(32)^{\frac{1}{5}}+\frac{1}{80}$

Using (i)
$$(33)^{\frac{1}{5}}\approx 2+\frac{1}{80}=\frac{160+1}{80}$$
$$(33)^{\frac{1}{5}}\approx 2.0125$$

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(iii) $\tan 46°$ 12302113

Solution:

$\tan 46°$

Let $f(x) = \tan x$ .....(i)

Differentiate w.r.t. 'x':

$$f'(x) = \frac{d}{dx}\tan x$$

$$f'(x) = \sec^2 x \text{ .....(ii)}$$

Close $x = 45°$ because calculating $\tan 45°$ is easy. We approximate $\tan 46°$

So, $x + \Delta x = 46°$ $\boxed{\because x = 45°}$

$45° + \Delta x = 46°$

$\Delta x = 46° - 45°$

$\Delta x = 1°$, $1° = \frac{\pi}{180} Radians$

Since $dx = \Delta x$

$$\boxed{dx = \frac{\pi}{180} rad.}$$

We know that

$dy = f'(x)dx$

Using Equation (ii),

$dy = \sec^2 x dx$

$dy = \sec^2(45°)\left(\frac{\pi}{180}\right) Radian$

$dy = \left(\sqrt{2}\right)^2 \cdot \frac{\pi}{180} Radian$

$dy = 2\left(\frac{\pi}{180}\right) Radian$

$dy = 0.0349$

We know that

$f(x + \Delta x) \approx f(x) + dy$

$f(45° + 1°) \approx f(45°) + 0.0349$

$f(46°) \approx f(45°) + 0.0349$

Using (i)

$\tan 46° \approx \tan 45° + 0.0349$

$\approx 1 + 0.0349$

$\tan 46° \approx 1.0349$

4. The side of a cube is measured to be 30cm with a maximum error of 0.14cm in its measurement. Find the maximum error in the calculated volume of the cube. 12302114

Solution:

Let x be the side of cube:

$x = 30cm,\ dx = \Delta x = 0.14cm$

Let 'V' be the volume of cube:

$V = x^3$

Differentiate w.r.t. 'x':

$$\frac{d}{dx}(V) = \frac{d}{dx}\left(x^3\right)$$

$$\frac{dV}{dx} = 3x^2$$

$dV = 3x^2 dx$

Substitute $x = 30cm,\ dx = 0.14cm$

$dV = 3(30)^2(0.14)$

$= 3(900)(0.14)$

$dV = 378cm^3$

Maximum error in the volume of cube is $378\,cm^3$.

5. The radius of a sphere is measured as 100 ± 0.5 cm and the volume is calculated from this measurement. Estimate the percentage error in the Volume calculation. 12302115

Solution:

Let volume of sphere $V = \frac{4}{3}\pi r^3$

Differentiate w.r.t. 'r'

$$\frac{dV}{dr} = \left(\frac{4}{3}\pi r^3\right)$$

$$\frac{dV}{dr} = \frac{4}{3}\pi\left(3r^2\right)$$

$$\frac{dV}{dr} = 4\pi r^2$$

$dV = 4\pi r^2 dr$

Now,

$$\frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^2}$$

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$$\frac{dV}{V}=\frac{3dr}{r} \ldots\ldots\ldots(i)$$

$$\text{Percentage error} = \left|\frac{dV}{V}\right|\times 100\%$$

$$= \left|\frac{3dr}{r}\right|\times 100\%$$

Given: $r = 100cm, dr = 0.5cm$

By putting the values;

$$\%\text{ error} = \left|3\times\frac{0.5}{100}\right|\times 100\%$$

$$= 3\times 0.005\times 100\%$$

$$\%\text{ error} = 1.5\%$$

Applying Extrema to Real-World Problems

To solve extrema problems in real-life scenarios, follow these steps:

1. **Understand the Problem** – Identify what needs to be optimized.
2. **Define Variables** – Assign variables to unknown quantities.
3. **Formulate the Function** – Write an equation representing the quantity to optimize.
4. **Find Constraints** – If applicable, express limitations.
5. **Find Critical Points** – Take the derivative and find critical points.
6. **Verify Extrema** – Use the second derivative test to confirm maxima/minima.
7. **Interpret the Solution** – Relate the mathematical answer back to the real-world context.

Example 30: A rectangular garden is to be fenced on three sides with 100 meters of fencing. The fourth side is along a wall and requires no fence. Find the dimensions that maximize the area and the maximum area. **12302116**

Solution:

Let the side opposite to the wall be x meters.

Let each side perpendicular to the wall by y meters.

Total length of the garden to be fenced:

$$x+2y=100$$

$$y=\frac{100-x}{2} \ldots\ldots\ldots(i)$$

Area of the garden $= A = xy$

Substituting the expression of y from (i):

$$A = x\left(\frac{100-x}{2}\right)$$

$$A=\frac{1}{2}\left(100x-x^2\right) \ldots\ldots\ldots(ii)$$

Differentiating w.r.t. 'x':

$$\frac{dA}{dx}=\frac{d}{dx}\left[\frac{1}{2}\left(100x-x^2\right)\right]$$

$$\frac{dA}{dx}=\frac{1}{2}\left(100-2x\right)$$

$$=\frac{1}{2}\times 100-\frac{1}{2}\times 2x$$

$$\frac{dA}{dx}=50-x \ldots\ldots\ldots(iii)$$

Put $\frac{dA}{dx}=0$:

$$50-x=0$$

$$x=50$$

Thus, $x=50$ is the critical point.

Differentiating equation (i) w.r.t. 'x':

$$\frac{d}{dx}\left(\frac{dA}{dx}\right)=\frac{d}{dx}\left(50-x\right)$$

$$\frac{d^2A}{dx^2}=0-1$$

$$\frac{d^2A}{dx^2}=-1$$

Since $\frac{d^2A}{dx^2}=-1<0,$ the area A is maximum at $x=50$ meters

Substitute $x=50$ into equations (i) and (ii):

$$y=\frac{100-50}{2}=25\,\text{meters and}$$

$$A=\frac{1}{2}\left(100\times 50-50^2\right)=1250\,m^2$$

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The maximum fenced area of the garden is $1250\,m^2$, which occurs when:

- The side opposite the wall is 50 meters.
- Each of the sides perpendicular to the wall is 25 meters.

Example 31: Find the dimensions of a rectangle of maximum area that can be inscribed under the curve $y = 4 - x^2$ and above the x-axis. 12302117

Solution:

Let one vertex of the rectangle on the parabola in first quadrant be $A(x, y)$.

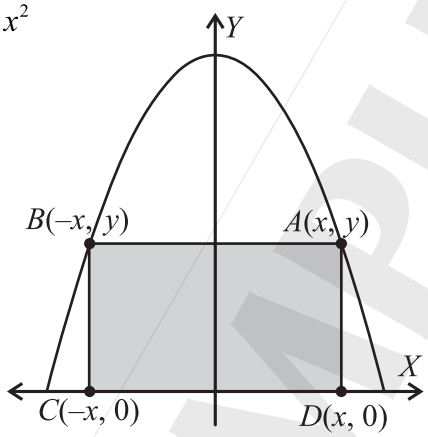
The other vertex in second quadrant is $B(-x, y)$.

Height of the rectangle $= y$

Width of the rectangle $= 2x$

Area of the rectangle: $A = 2xy$ .....(i)

The vertex $A(x, y)$ of the rectangle satisfies $y = 4 - x^2$



Substitution this expression of y into equation (i):

$$A = 2x(4 - x^2)$$
$$= 2(4x - x^3)$$

Differentiate with respect to x:

$$\frac{dA}{dx} = \frac{d}{dx}\left[2(4x - x^2)\right]$$
$$= 2\left[\frac{d}{dx}(4x) - \frac{d}{dx}(x^3)\right]$$
$$\frac{dA}{dx} = 2(4 - 3x^2) \text{ .....(ii)}$$

For critical points:

Put $\frac{dA}{dx} = 0$:

$$2(4 - 3x^2) = 0 \Rightarrow 3x^2 = 4 \Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \frac{2}{\sqrt{3}}$$

We consider only positive root since x in first quadrant is positive.

Differentiating $\frac{dA}{dx} = 2(4 - 3x^2)$ w.r.t. x:

$$\frac{d^2A}{dx^2} = 2(-6x) = -12x$$

At $x = \frac{2}{\sqrt{3}}$:

$$\frac{d^2A}{dx^2} = -12\left(\frac{2}{\sqrt{3}}\right) = -\frac{24}{\sqrt{3}} < 0$$

Negative second derivative implies that area is maximum at $x = \frac{2}{\sqrt{3}}$.

Substitute $x = \frac{2}{\sqrt{3}}$ into $y = 4 - x^2$:

$$y = 4 - \left(\frac{2}{\sqrt{3}}\right)^2 = 4 - \frac{4}{3} = \frac{8}{3}$$

The dimensions of the rectangle of maximum area are:

- Width $= 2x = \frac{4}{\sqrt{3}}$ units
- Height $= \frac{8}{3}$ units

Example 32: A cylindrical can must hold 500 cm^3 of liquid. Find the radius and height that minimizes the material cost.

Solution: 12302118

The cost of the material will be minimized if the surface area of the cylinder is minimized. Our objective is to minimize the surface area of the cylinder.

Let:

- Radius of the base $= r\,cm$
- Height of the cylinder $= h\,cm$

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Volume of the cylinder:

$$V = \pi r^2 h = 500\,cm^3$$

$$h = \frac{500}{\pi r^2} \ldots\ldots\ldots\text{(i)}$$

Surface area of cylinder:

- Surface area of two circular ends $= 2\pi r^2$
- Curved surface area $= 2\pi rh$

Total surface area of the cylinder:

$$S = 2\pi r^2 + 2\pi rh \ldots\ldots\ldots\text{(ii)}$$

Substitute the expression of h from equation (i):

$$S = 2\pi r^2 + 2\pi r \times \frac{500}{\pi r^2} = 2\pi r^2 + \frac{1000}{r}$$

Differentiate 'S' w.r.t. 'r':

$$\frac{dS}{dr} = 4\pi r - \frac{1000}{r^2} \ldots\ldots\ldots\text{(iii)}$$

For critical points:

Set $\frac{dS}{dr} = 0$:

$$4\pi r - \frac{1000}{r^2} = 0 \Rightarrow 4\pi r = \frac{1000}{r^2} \Rightarrow r^3 = \frac{1000}{4\pi}$$

$$r = \left(\frac{250}{\pi}\right)^{1/3}$$

Differentiate equation (iii) w.r.t. r:

$$\frac{d^2S}{dr^2} = 4\pi + \frac{2000}{r^3}$$

Substitute $r = \left(\frac{250}{\pi}\right)^{1/3}$ into $\frac{d^2S}{dr^2}$:

$$\frac{d^2S}{dr^2} = 4\pi + \frac{2000}{\left[\left(\frac{250}{\pi}\right)^{1/3}\right]^3} > 0$$

Since $\frac{d^2S}{dr^2} => 0, S$ is minimum at $r = \left(\frac{250}{\pi}\right)^{\frac{1}{3}}$.

Substitute $r = \left(\frac{250}{\pi}\right)^{\frac{1}{3}}$ into $h = \frac{500}{\pi r^2}$:

$$h = \frac{500}{\pi\left[\left(\frac{250}{\pi}\right)^{\frac{1}{3}}\right]^2} = \frac{500}{(\pi)^{\frac{1}{3}}(250)^{\frac{2}{3}}}$$

The material cost is minimized when:

- Radius $= r = \left(\frac{250}{\pi}\right)^{\frac{1}{3}} cm$
- Height $= h = \frac{500}{(\pi)^{\frac{1}{3}}(250)^{\frac{2}{3}}} cm$

Example 33: A car moves along the parabolic road $y = 4 - x^2$. A mobile tower is at the origin. Find the position of the car where the signal strength is strongest.

Solution: 12302119

Signal strength is strongest when the distance between the car and the tower is minimum.

Let the position of the car be $P(x, y)$ on the parabola $y = 4 - x^2$

The distance D of the car form the tower at origin $(0,0)$ is:

$$D = \sqrt{x^2 + y^2} \ldots\ldots\ldots\text{(i)}$$

Since the car lies on the parabola $y = 4 - x^2$

$$\Rightarrow x^2 = 4 - y \ldots\ldots\ldots\text{(ii)}$$

Substitute $x^2 = 4 - y$ into equation (i):

$$D = \sqrt{4 - y + y^2}$$

Since minimizing D is equivalent to minimizing D^2:

$$D^2 = 4 - y + y^2$$

Let $D^2 = f(y)$:

$$f(y) = y^2 - y + 4$$

Differentiate with respect to y:

$$\frac{df}{dy} = 2y - 1 \ldots\ldots\ldots\text{(iii)}$$

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For critical points:

Set $\frac{df}{dy}=0$:

$$2y-1=0$$
$$y=\frac{1}{2}$$

Differentiate equation (iii) with respect to x:

Since second derivative is positive f is minimum at $y=\frac{1}{2}$.

Substitute $y=\frac{1}{2}$ into $x^2=4-y$:

$$x^2=4-\frac{1}{2}=\frac{7}{2}$$
$$x=\pm\sqrt{\frac{7}{2}}$$

The positions of the car where signal strength are $\left(\sqrt{\frac{7}{2}},\frac{1}{2}\right)$ and $\left(-\sqrt{\frac{7}{2}},\frac{1}{2}\right)$.

Example 34: A software company is designing a data processing algorithm for analyzing batches of customer transactions. The runtime $T(n)$ in milliseconds depends on the size of the batch *n* (the number of transactions processed together) and is modeled by the equation:

$$T(n)=n^3-12n^2+36n$$

- **The term n^3 reflects the time complexity of a strong step in algorithm.**
- **The term $-12n^2$ accounts for optimizations that reduce processing time for moderate batch sizes.**
- **The term $36n$ represents overhead that scales linearly with batch size.**

The company wants to find the optimal batch size *n* that minimizes the algorithm's total runtime. **12302120**

Solution:

The runtime of the algorithm is:

$$T(n)=n^3-12n^2+36n \ldots\ldots\ldots\text{(i)}$$

We need to find the input size n (the number of transactions processed together) that minimizes the runtime $T(n)$.

Differentiate equation (i) with respect to n:

$$\frac{dT}{dn}=\frac{d}{dn}\left(n^3-12n^2+36n\right)$$
$$\frac{dT}{dn}=3n^2-24n+36 \ldots\ldots\ldots\text{(ii)}$$

To find critical points set $\frac{dT}{dn}=0$:

$$3n^2-24n+36=0$$
$$3\left(n^2-8n+12\right)=0$$
$$n^2-8n+12=0 \qquad \boxed{\because 3\neq 0}$$
$$(n-2)(n-6)=0$$
$$n-2=0,\ n-6=0$$

Thus, the critical points are:

$n=2$ and $n=6$

Differentiation equation (ii) with respect to n:

$$\frac{dT}{dn}=\frac{d}{dn}\left(3n^2-24n+36\right)$$
$$\frac{d^2T}{dn^2}=6n-24 \ldots\ldots\ldots\text{(iii)}$$

Substitute $n=2$ and $n=6$ into equation (iii)

$\frac{d^2T}{dn^2}=6\times 2-24=-12<0$ and

$\frac{d^2T}{dn^2}=6\times 6-24=12>0$

Since $\frac{d^2T}{dn^2}<0$ at $n=2$, the function $T(n)$ has a local maximum at $n=2$.

Since $\frac{d^2T}{dn^2}>0$ at $n=6$, the function $T(n)$ has a local minimum at $n=6$.

The runtime is minimized when the input size is $n=6$.

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EXERCISE 2.7

1. A window consists of a rectangle surmounted by a semicircular opening. If the total perimeter is 21 meters, find the dimensions that maximize the area.

Solution: 12302121

Perimeter $= P = 21m$

Let the width of the rectangle be $2r$, so the radius of the semi-circle is r and let height of the rectangle be $2h$.

Then perimeter will be:

$$P = 2h + 2r + \pi r$$
$$21 = 2h + 2r + \pi r$$
$$2h = 21 - 2r - \pi r$$
$$h = \frac{21-(2+\pi)r}{2} \quad \ldots\ldots\ldots(i)$$

Total area A is the area of the rectangle plus area of semi-circle is

$$A = 2rh + \frac{1}{2}\pi r^2 \quad \ldots\ldots\ldots(ii)$$

$$A = 2r\left[\frac{21-(2+\pi)r}{2}\right] + \frac{1}{2}\pi r^2 \text{ using (i)}$$

$$A = r\left[21-(2+\pi)r\right] + \frac{1}{2}\pi r^2$$

$$A = 21r - (2+\pi)r^2 + \frac{1}{2}\pi r^2$$

$$= 21r - \left(2+\pi-\frac{\pi}{2}\right)r^2$$

$$= 21r - \left(2+\frac{\pi}{2}\right)r^2$$

$$A = 21r - \left(\frac{4+\pi}{2}\right)r^2$$

Differentiate w.r.t. 'x',

$$\frac{dA}{dr} = \frac{d}{dr}\left[21r - \left(\frac{4+\pi}{2}\right)r^2\right]$$

$$\frac{dA}{dr} = \frac{d}{dr}(21r) - \frac{d}{dr}\left(\frac{4+\pi}{2}\right)r^2$$

$$\frac{dA}{dr} = 21 - \left(\frac{4+\pi}{2}\right)2r$$

$$\frac{dA}{dr} = 21 - (4+\pi)r$$

For stationary points:

Put $\frac{dA}{dr} = 0$

$$21 - (4+\pi)r = 0$$
$$21 = (4+\pi)r$$
$$\boxed{\Rightarrow r = \frac{21}{4t\pi}}$$

$r = \frac{21}{4+\pi}$ put in equation (i)

$$h = \frac{21-(2+\pi)\left(\frac{21}{4+\pi}\right)}{2}$$

$$h = \frac{\frac{21(4+\pi)-21(2+\pi)}{(4+\pi)}}{2}$$

$$h = \frac{21(4+\pi-2-\pi)}{2(4+\pi)}$$

$$h = \frac{2(21)}{2(4+\pi)}$$

$$h = \frac{21}{(4+\pi)}$$

Height $= h = \frac{21}{4+\pi} \approx 2.94\,m$

Width $= 2r = \frac{42}{4+\pi} \approx 5.88\,m$

2. A rectangular sheet of metal with dimension 40 cm by 25 cm is to be made into an open box by cutting equal squares from the corners and folding up the sides. Find the size of the squares that maximize the volume.

Solution: 12302122

Let the side of the square cut from each side is x.

Length $= (40-2x)\,cm$

Width $= (25-2x)\,cm$

Height $= x\,cm$

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Then volume becomes;

$V = length \times width \times height$

$V = (40-2x)(25-2x)(x)$

$V = (1000-80x-50x+4x^2)(x)$

$V = (1000-130x+4x^2)x$

$V = 1000x-130x^2+4x^3$

Differentiate w.r.t. 'x':

$$\frac{dV}{dx} = \frac{d}{dx}(1000x-130x^2+4x^3)$$

$$\frac{dV}{dx} = \frac{d}{dx}(1000x)-\frac{d}{dx}(130x^2)+\frac{d}{dx}(4x^3)$$

$$\frac{dV}{dx} = 1000-260x+12x^2 \ldots\ldots\ldots(i)$$

For stationary point:

Put $\frac{dV}{dx} = 0$

$12x^2-260x+1000 = 0$

$4(3x^2-65x+250) = 0$

$3x^2-65x+250 = 0$

Using quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Here $a = 3, b = -65, c = 250$

Putting values;

$$x = \frac{-(-65) \pm \sqrt{(-65)^2-4(3)(250)}}{2(3)}$$

$$x = \frac{65 \pm \sqrt{4225-3000}}{6}$$

$$x = \frac{65 \pm \sqrt{1225}}{6}$$

$$x = \frac{65 \pm 35}{6}$$

$$x = \frac{65+35}{6}, \quad x = \frac{65-35}{6}$$

$$x = \frac{100}{6}, \quad x = \frac{30}{6}$$

$$x = \frac{50}{3}, \quad x = 5$$

which are critical points.

Again differentiate w.r.t. 'x':

$$\frac{d}{dx}\left(\frac{dV}{dx}\right) = \frac{d}{dx}(1000-260x+12x^2)$$

$$\frac{d^2V}{dx^2} = 0-260+24x$$

$$\frac{d^2V}{dx^2} = 24x-260$$

At $x = 5$

$$\frac{d^2V}{dx^2} = 24(5)-260$$

$$= 120-260$$

$$= -140 < 0$$

Volume will be maximum at $x = 5$.

At $x = \frac{50}{3}$

$$\frac{d^2A}{dx^2} = 24\left(\frac{50}{3}\right)-260$$

$$= \frac{1200-780}{3}$$

$$= \frac{420}{3} = 140 > 0$$

Volume will be minimum.

As volume will be maximum at $x = 5$.

So,

The size of square cut is $5\,cm$

3. A closed cylindrical tank is to be constructed to hold 54π liters of water. Find the dimensions that minimize the construction cost of the tank. 12302123

Solution:

Let height of cylinder $= h$ meter

Radius $= r$ meter

Volume of cylinder $V = \pi r^2 h$

Volume $= V = 54\,m$

So, $54\pi = \pi r^2 h$

$54 = r^2 h$

$$h = \frac{54}{r^2} \ldots\ldots\ldots(i)$$

Total surface area: $A = 2\pi r^2 + 2\pi rh \ldots\ldots(ii)$

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$$A = 2\pi r^2 + 2\pi r\left(\frac{54}{r^2}\right) \text{ using (i)}$$

$$A = 2\pi r^2 + \frac{108\pi}{r}$$

Differentiate w.r.t. 'r'

$$\frac{dA}{dr} = \frac{d}{dr}\left(2\pi r^2 + \frac{108\pi}{r}\right)$$

$$= \frac{d}{dr}\left(2\pi r^2\right) + \frac{d}{dr}\left(\frac{108\pi}{r}\right)$$

$$\frac{dA}{dr} = 2\pi\left(2r\right) + 108\pi\left(-r^{-2}\right)$$

$$\frac{dA}{dr} = 4\pi r - \frac{108\pi}{r^2} \quad \ldots\ldots\ldots\text{(iii)}$$

For stationary points:

Put $\frac{dA}{dr} = 0$

$$4\pi r - \frac{108}{r^2}\pi = 0$$

$$4\pi r^3 - 108\pi = 0$$

$$4\pi r^3 = 108\pi$$

$$r^3 = \frac{108\pi}{4\pi}$$

$$r^3 = 27$$

$$r^3 = 3^3$$

$$\boxed{r = 3}$$

Again differentiate w.r.t. 'r':

$$\frac{d^2A}{dr^2} = \frac{d}{dr}\left(4\pi r - \frac{108\pi}{r^2}\right)$$

$$= 4\pi - 108\pi\left(-2r^{-3}\right)$$

$$= 4\pi + \frac{216\pi}{r^3}$$

At $r = 3$

$$\frac{d^2A}{dr^2} = 4\pi + \frac{216\pi}{(3)^3}$$

$$\frac{d^2A}{dr^2} = 4\pi + 8\pi$$

$$\frac{d^2A}{dr^2} = 12\pi > 0$$

Which show height is minimum at $r = 3$

So,

$$h = \frac{54}{r^2} = \frac{54}{(3)^2} = \frac{54}{9} = 6\ dm$$

Radius $= r = 3\ dm$

4. A wire of 20 meters is cut into two pieces. One piece is bent into a square and the other into a circle. How should the wire be cut to maximize the total are enclosed? **12302124**

Solution:

Let the length of wire used for the square be x, so the wire left the circle is $20 - x$

Side of square $= \frac{x}{4}$

Area of square $= \left(\frac{x}{4}\right)^2 = \frac{x^2}{16} \quad \ldots\ldots\ldots\text{(i)}$

Circumference of circle $= C = 2\pi r = 20 - x$

$$r = \frac{20 - x}{2\pi} \quad \ldots\ldots\ldots\text{(ii)}$$

Area of circle $A = \pi r^2$

$$A = \pi\left(\frac{20 - x}{2\pi}\right)^2 \text{ using (ii)}$$

$$A = \frac{(20 - x)^2}{4\pi}$$

Now total area = area of square + area of circle

$$A = \frac{x^2}{16} + \frac{(20 - x)^2}{4\pi}$$

Differentiate w.r.t. 'x':

$$\frac{dA}{dx} = \frac{d}{dx}\left[\frac{x^2}{16} + \frac{(20 - x)^2}{4\pi}\right]$$

$$= \frac{d}{dx}\left(\frac{x^2}{16}\right) + \frac{d}{dx}\frac{(20 - x)^2}{4\pi}$$

$$= \frac{1}{16}\left(2x\right) + \frac{1}{4\pi} \cdot 2\left(20 - x\right)\left(0 - 1\right)$$

$$\frac{dA}{dx} = \frac{x}{8} - \frac{20 - x}{2\pi} \quad \ldots\ldots\ldots\text{(iii)}$$

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For stationary points:

Put $\frac{dA}{dx}=0$

$$\frac{x}{8}-\frac{(20-x)}{2\pi}=0$$
$$\frac{x}{8}=\frac{(20-x)}{2\pi}$$
$$2\pi x=8(20-x)$$
$$\pi x=4(20-x)$$
$$\pi x=80-4x$$
$$\pi x+4x=80$$
$$x(\pi+4)=80$$
$$x=\frac{80}{\pi+4}$$

Differentiate (iii) w.r.t 'x' again,

$$\frac{d^2A}{dx^2}=\frac{d}{dx}\left(\frac{x}{8}-\frac{20-x}{2\pi}\right)$$
$$A''(x)=\frac{1}{8}+\frac{1}{2\pi}>0$$

which show minimum

Square write bent into: $x=\frac{80}{\pi+4}m$

Circle write bent into:

$$=20-\left(\frac{80}{\pi+4}\right)$$
$$=\frac{20(\pi+4)-80}{\pi+4}$$
$$=\frac{20\pi+80-80}{\pi+4}$$
$$=\frac{20\pi}{\pi+4}$$

5. A team of environmental scientists is monitoring the oxygen level in a lake over time to ensure the water quality remains suitable for aquatic life. The oxygen level in the lake, measured in milligrams per liter (mg/L), is modeled by the equation:

$$O(t)=t^3-18t^2+81t+100$$

Where $O(t)$ is the oxygen level at time *t* (in days after monitoring began).

Scientists want to determine when during the monitoring period the oxygen level is at its lowest, as low oxygen levels can be harmful to fish and other aquatic organisms.

Find the time *t* when the oxygen level is at its minimum. **12302125**

Solution:

As given $O(t)=t^3-18t^2+81t+100$

Differentiate w.r.t. 't'

$$O'(t)=\frac{d}{dt}\left[t^3-18t^2+81t+100\right]$$
$$O'(t)=3t^2-36t+81.....\text{(i)}$$

For stationary point $O'(t)=0$

$$3t^2-36t+81=0$$
$$t^2-12t+27=0$$
$$t^2-9t-3t+27=0$$
$$t(t-9)-3(t-9)=0$$
$$(t-3)(t-9)=0$$
$$t=3,9$$

Again Differentiate (i) w.r.t. 't'

$$O'(t)=\frac{d}{dt}(3t^2-36t+81)$$
$$O''(t)=6t-36$$

At $t=3$

$$O''(3)=6(3)-36$$
$$=-18<0\,\text{(Maximum)}$$

At $t=9$

$$O''(3)=6(9)-36$$
$$=54-36$$
$$=18>0\,\text{(Minimum)}$$

Conclusion: Time at Minimum oxygen level at $t=9$ days.

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6. The perimeter of a triangle is 16 cm. If one side is of length 6 cm, what are the lengths of the other sides for maximum area of the triangle. 12302126

Solution:

Perimeter $= P = 16cm$

Let the sides of the triangle be:

$a = 6,\ b = x$

Perimeter $= 16$

$a + b + c = 16$

$6 + x + c = 16$

$c = 16 - 6 - x$

$\boxed{c = 16 - x}$

As we know that: $2s = a + b + c$

$2s = 16$

$\boxed{s = 8}$

By Hero's formula;

$A = \sqrt{s(s-a)(s-b)(s-c)}$

$A = \sqrt{8(8-6)(8-x)(8-(10-x))}$

$A = \sqrt{16(8-x)(x-2)}$

Maximize expression inside square root.

i.e.

$$f(x) = 16(8-x)(x-2)$$
$$f(x) = 16(8x - 16 - x^2 + 2x)$$
$$f(x) = 16(-x^2 + 10x - 16)$$

Differentiate w.r.t 'x':

$$f'(x) = \frac{d}{dx}\left[16(-x^2 + 10x - 16)\right]$$
$$f'(x) = 16(-2x + 10) \ldots\ldots\ldots(i)$$

For stationary points:

For $f'(x) = 0$

$$16(-2x + 10) = 0$$
$$-2x + 10 = 0 \quad \boxed{\because 16 \neq 0}$$
$$2x = 10$$
$$x = 5$$

Again Differentiate (i) w.r.t. 'x':

$$f''(x) = \frac{d}{dx}\left[16(-2x + 10)\right]$$
$$f''(x) = 16(-2) = -32 < 0$$

which shows area is maximum.

Now the sides are:

$a = 6cm,\ b = 5cm,$

$c = 10 - x$

$c = 10 - 5$

$c = 5cm$

So the lengths of other two sides are $b = 5cm, c = 5cm.$

7. A poster must have 200 cm^2 printed area with margins of 2 cm on top/bottom and 3 cm on sides. Find poster dimensions to minimize paper used. 12302127

Solution:

Let the width and height of the printed area be x and y respectively.

Printed area; $xy = 200$

$$y = \frac{200}{x} \ldots\ldots\ldots(i)$$

Now total width $= x + 6$

Total height $= y + 4$

Total height $= \frac{200}{x} + 4$ using (i)

$$A = (x+6)\left(\frac{200}{x} + 4\right)$$

$$A = 200 + 4x + \frac{1200}{x} + 24$$

Differentiate w.r.t. 'x':

$$\frac{dA}{dx} = \frac{d}{dx}\left(200 + 4x + \frac{1200}{x} + 24\right)$$

$$\frac{dA}{dx} = \frac{d}{dx}(200) + \frac{d}{dx}(4x) + \frac{d}{dx}\left(\frac{1200}{x}\right) + \frac{d}{dx}(24)$$

$$\frac{dA}{dx} = 0 + 4 - \frac{1200}{x^2} + 0$$

$$\frac{dA}{dx} = 4 - \frac{1200}{x^2} \ldots\ldots\ldots(iii)$$

For stationary points:

Put $\frac{dA}{dx} = 0$

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$$4-\frac{1200}{x^2}=0$$

$$4=\frac{1200}{x^2}$$

$$4x^2=1200$$

$$x^2=300$$

$$x^2=100\times 3$$

$$\sqrt{x^2}=\sqrt{100\times 3}$$

$$x=10\sqrt{3}$$

Put in equation (i) $y=\frac{200}{10\sqrt{3}}=\frac{20}{\sqrt{3}}$

Differentiate (iii) again w.r.t. 'x':

$$\frac{d^2A}{dx^2}=\frac{d}{dx}\left(4-\frac{1200}{x^2}\right)$$

$$=\frac{d}{dx}(4)-\frac{d}{dx}\left(\frac{1200}{x^2}\right)$$

$$=0+\frac{2400}{x^3}$$

$$\frac{d^2A}{dx^2}=\frac{2400}{x^3}$$

At $x=10\sqrt{3}$

$$\frac{d^2A}{dx^2}=\frac{2400}{\left(10\sqrt{3}\right)^3}>0$$

So, Area is minimum.

Now total width $=x+6=10\sqrt{3}+6$

$$=10(1.73)+6$$

$$=17.3+6$$

$$=23.32\,cm$$

Total height $=y+4=\frac{20}{\sqrt{3}}+4$

$$=11.55+4$$

$$=15.55\,cm$$

8. Find two positive integers whose sum is 12 and the product of one with the square of the other will be maximum.

Solution: **12302128**

Let one positive integer is x than the other positive integer will be $12-x$.

$\because f(x)=x(12-x)^2$ (Given)

$$f(x)=x\left(144-24x+x^2\right)$$

$$f(x)=144x-24x^2+x^3 \ldots\ldots\ldots\text{(i)}$$

Differentiate w.r.t. 'x':

$$f'(x)=\frac{d}{dx}\left[144x-24x^2+x^3\right]$$

$$=\frac{d}{dx}(144x)-\frac{d}{dx}\left(24x^2\right)+\frac{d}{dx}\left(x^3\right)$$

$$f'(x)=144-48x+3x^2 \ldots\ldots\ldots\text{(ii)}$$

For stationary points:

Put $f'(x)=0$

$$3x^2-48x+144=0$$

$$3\left(x^2 16x+48\right)=0$$

$$x^2-16x+48=0$$

$$x^2-12x-4x+48=0$$

$$x(x-12)-4(x-12)=0$$

$$(x-12)(x-4)=0$$

$$x-4=0,\ x-12=0$$

$$x=4,\ x=12\ (\text{neglect})$$

For maximum area;

Differentiate (ii) again w.r.t. 'x':

$$f''(x)=\frac{d}{dx}\left(144-48x+3x^2\right)$$

$$f''(x)=0-48+6x$$

$$f''(x)=6x-48$$

At $x=4$

$$f''(x)=6(4)-48$$

$$=24-48$$

$$=-24<0$$

which shows maximum.

So, first integer $=x=4$

2^{nd} integer $=12-x=12-4=8$

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ADDITIONAL MULTIPLE CHOICE QUESTIONS

Review of Differentiation Basics

1. Derivative of the function $y = f(x)$ is denoted by: 12302129

(a) $\frac{dy}{dx}$ (b) $\frac{dy}{dx}$ or $f'(x)$

(c) dy (d) $cf(x)$

2. $\frac{d}{dx}(x^n) =$ _____ 12302130

(a) $\frac{x^n}{n}$ (b) nx^n

(c) nx^{n-1} (d) nx^{n+1}

3. $\frac{d}{dx}(c) =$ _____ 12302131

(a) $\frac{1}{2}$ (b) 3

(c) ∞ (d) 0

4. $\frac{d}{dx}(cu) =$ _____ 12302132

(a) cu' (b) $c\frac{du}{dx}$

(c) c' (d) Both 'a' and 'b'

5. According to Sum Rule $\frac{d}{dx}(u+v) =$ _____ 12302133

(a) $u'v + uv'$ (b) $u' - v'$

(c) $u' + v'$ (d) $\frac{u'v - uv'}{v^2}$

6. The product rule for differentiation is:

(a) $\frac{d}{dx}(uv) = u' + v'$ 12302134

(b) $\frac{d}{dx}(uv) = uv' + u'v$

(c) $\frac{d}{dx}(uv) = uv' - u'v$

(d) $\frac{d}{dx}(cu) = cu'$

7. The quotient rule is used to differentiate _____ of two functions. 12302135

(a) Sum (b) Product

(c) Division (d) Difference

8. $\frac{d}{dx}\left(\frac{u}{v}\right) =$ _____ 12302136

(a) $\frac{uv' + vu^2}{v^2}$ (b) $\frac{u'v - uv'}{v^2}, v(x) \neq 0$

(c) $u'v + vu'$ (d) $\frac{u' - v'}{v}$

9. In quotient rule what condition must be true for v? 12302137

(a) $v(x) = 0$

(b) $v(x) \neq 0$, for all x in the domain

(c) $v'(x) = 0$

(d) $v(x) = 1$

10. $\frac{d}{dx}(u - v)$ **is equal to:** 12302138

(a) $u' + v'$ (b) $u - v$

(c) $u' - v'$ (d) $u'v'$

11. If $y = x\sin x$ which rule should be used to differentiate it? 12302139

(a) Sum rule (b) Product rule

(c) Quotient rule (d) None of these

Derivative of Trigonometric Function

12. While finding derivative of trigonometric function, we assume that the angle is measured in _____.

(a) Degree (b) Radian 12302140

(c) Minutes (d) Seconds

13. $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} =$ _____, θ **is measured in radian.** 12302141

(a) 0 (b) 1

(c) 2 (d) ∞

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14. $\frac{d}{dx}(\sin x)$ **is equal to:** 12302142

(a) $\sin x$ (b) $\cos x$
(c) $-\sin x$ (d) $-\cos x$

15. $\frac{d}{dx}(\cos x)$ **is equal to:** 12302143

(a) $\sin x$ (b) $\cos x$
(c) $-\sin x$ (d) $-\cos x$

16. $\frac{d}{dx}(\tan x) =$ _____ 12302144

(a) $\sec^2 x$ (b) $-\sec^2 x$
(c) $-\tan x$ (d) $\tan x$

17. $\frac{d}{dx}(\operatorname{cosec} x)$ **is equal to:** 12302145

(a) $\operatorname{cosec} x \cot x$ (b) $-\operatorname{cosec} x \cot x$
(c) $\cot x$ (d) $-\cot x$

18. $\frac{d}{dx}(\sec x)$ 12302146

(a) $\tan x$ (b) $\cos x$
(c) $\sec x \tan x$ (d) $-\sec x \tan x$

The Chain Rule

19. If $g: X \to Y$ **and** $f: Y \to Z$ **then composition of** fog **is defined as:**

(a) $fog(x) = g(f(x))$ 12302147
(b) $fog(x) = f(g(x))$
(c) $fog(x) = f(x) + g(x)$
(d) $fog(x) = f(x) - g(x)$

20. If $f(u)$ **is differentiable at** $u = g(x)$ **and** $g(x)$ **is differentiable at** x **then** $fog(x)$ **is:** 12302148

(a) Not differentiable
(b) Differentiable at 'x'
(c) Differentiable only at u
(d) Always constant

21. The chain rule in Leibniz's notation is given by: 12302149

(a) $\frac{dy}{dx} = \frac{dy}{dx} + \frac{du}{dx}$ (b) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
(c) $\frac{dy}{dx} = \frac{dx}{dy} \cdot \frac{dx}{du}$ (d) $\frac{dy}{dx} = \frac{du}{dy} \cdot \frac{dx}{du}$

22. For a function represented by parametric equations $x = f(t)$ **and** $y = g(t)$, **with '**f**'bijective** $\frac{dy}{dx}$ **is equal to:** 12302150

(a) $\frac{dy}{dt} \cdot \frac{dx}{dt}$ (b) $\frac{dx}{dt} \cdot \frac{dt}{dy}$
(c) $\frac{dy}{dt} \cdot \frac{dt}{dx}$ (d) $\frac{dt}{dy} \cdot \frac{dt}{dx}$

23. In parametric differentiation, the formula $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$ **is derived using ___ rule.** 12302151

(a) Product (b) Quotient
(c) Chain (d) None of these

24. If $x = a\sin t$ **and** $y = a\cos t$ **then** $\frac{dy}{dx}$ **is equation to:** 12302152

(a) $\cos t$ (b) $\cot t$
(c) $\tan t$ (d) $-\tan t$

25. In implicit differentiation of $F(x, y) = 0$ **with respect to '**x**', how is** y **treated?** 12302153

(a) As a constant
(b) As an independent variable
(c) As a differentiable function of 'x'
(d) As zero

26. During implicit differentiation, which rule is used to differentiate terms that contain the dependent variable y **?** 12302154

(a) Product rule (b) Quotient rule
(c) Chain rule (d) Power rule

Derivative of Inverse Trigonometric Function

27. $\frac{d}{dx}(\sin^{-1} x)$ **is equal to:** 12302155

(a) $\frac{1}{\sqrt{1-x^2}}$ (b) $\frac{-1}{\sqrt{1-x^2}}$
(c) $\frac{1}{x^2+1}$ (d) $\frac{1}{x+1}$

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28. $\frac{d}{dx}\left(\cos^{-1}x\right)=$ _____ **12302155(a)**

(a) $\frac{1}{\sqrt{1-x^2}}$ (b) $\frac{-1}{\sqrt{1-x^2}}$

(c) $\frac{1}{x^2+1}$ (d) $\frac{-1}{x^2+1}$

29. $\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right)=$ _____ **12302156**

(a) $\frac{1}{\sqrt{a^2-x^2}}$ (b) $\frac{-1}{\sqrt{a^2-x^2}}$

(c) $\frac{-1}{\sqrt{x^2-a^2}}$ (d) $\frac{1}{\sqrt{x^2+a^2}}$

30. $\frac{d}{dx}\left(\tan^{-1}x\right)=$ _____ **12302157**

(a) $\frac{1}{\sqrt{1-x^2}}$ (b) $\frac{1}{|x|\sqrt{x^2-1}}$

(c) $\frac{1}{1+x^2}$ (d) $\frac{-1}{1+x^2}$

31. $\frac{d}{dx}\left(\text{cosec}^{-1}x\right)=$ _____ **12302158**

(a) $\frac{-1}{|x|\sqrt{x^2-1}}$ (b) $\frac{1}{|x|\sqrt{x^2-1}}$

(c) $\frac{1}{1+x^2}$ (d) $\frac{1}{\sqrt{1-x^2}}$

32. $\frac{d}{dx}\left(\sec^{-1}x\right)=$ _____ **12302159**

(a) $\frac{-1}{|x|\sqrt{x^2-1}}$ (b) $\frac{1}{|x|\sqrt{x^2-1}}$

(c) $\frac{1}{1+x^2}$ (d) $\frac{1}{\sqrt{1+x^2}}$

33. $\frac{d}{dx}\left(\cot^{-1}x\right)=$ _____ **12302160**

(a) $\frac{1}{1+x^2}$ (b) $\frac{1}{1-x^2}$

(c) $\frac{1}{x^2-1}$ (d) $\frac{-1}{1+x^2}$

34. $\frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right)=$ _____ **12302161**

(a) $\frac{1}{\sqrt{a^2-x^2}}$ (b) $\frac{-1}{\sqrt{a^2-x^2}}$

(c) $\frac{1}{\sqrt{a^2+x^2}}$ (d) $\frac{1}{\sqrt{x^2-a^2}}$

Derivative of Exponential Function

35. A function defined by $f(x)=a^x$, where $a>0,\ a\neq 1$ is called _____ function. **12302162(a)**

(a) Radical (b) Quadratic

(c) Exponential (d) Linear

36. The natural exponential function is: **12302162**

(a) $f(x)=e^x$ (b) $f(x)=2x+3$

(c) $f(x)=\log x$ (d) $f(x)=\sqrt{x+2}$

37. $\lim_{h\to 0}\frac{a^h-1}{h}=$ _____ **12302163**

(a) $\ln 1$ (b) $\ln h$

(c) $\ln a$ (d) $2\ln a$

38. $\ln e$ **is equal to:** **12302164**

(a) 0 (b) 1

(c) 2 (d) ∞

39. $\frac{d}{dx}\left(e^x\right)=$ _____ **12302165**

(a) xe^x (b) $2e^x$

(c) e^x (d) $\frac{e^x}{2}$

40. $\frac{d}{dx}\left(a^x\right)$ **is equal to:** **12302166**

(a) a^x (b) $a^x\ln a$

(c) $\ln a^x$ (d) $a\ln a^x$

41. $\frac{d}{dx}\left(a^{x^3}\right)$ **is equal to:** **12302167**

(a) $3x^2a^{x^3}\ln a$ (b) $3x\ln a^x$

(c) $3x\ln a$ (d) $\ln a$

42. If $y=e^{\sin^2 x}$, **then** $\frac{dy}{dx}=$ _____ **12302168**

(a) $2\sin x e^{\sin^2 x}$ (b) $e^{\sin x}\cos x$

(c) $2\sin x\cos x e^{\sin^2 x}$ (d) $2e^{\sin x}$

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43. If $y = 5^x$ then $\frac{dy}{dx}$ is equal to: 12302169

(a) 5^x (b) $5^x \ln 5$

(c) $\ln 5^x$ (d) $x \ln 5$

44. $\frac{d}{dx}\left(e^{\cos x}\right) =$ _____ 12302170

(a) $-\sin x e^{\cos x}$ (b) $\sin x e^{\cos x}$

(c) $\cos x e^{\cos x}$ (d) $-\cos x e^{\cos x}$

Derivative of logarithm Functions

45. If $a^y = x$, where $a > 0$ and $a \neq 1$ then y is called the: 12302171

(a) Exponent of x to the base 'a'

(b) Logarithm of x to the base 'a'

(c) Power of x

(d) Antilog of x

46. If $y = \log_e x = \ln x$ is called ____ logarithm of x. 12302172

(a) The natural (b) Common

(c) Anti (d) All of these

47. If $y = \ln x$ then $\frac{dy}{dx}$ is equal to: 12302173

(a) x (b) $\frac{1}{x}$

(c) $\frac{1}{x^2}$ (d) x^2

48. $\lim_{h \to 0}\left(1 + \frac{h}{a}\right)^{\frac{a}{h}} =$ _____ 12302174

(a) 1 (b) 0

(c) e (d) ∞

49. If $y = \log_a x$ then $\frac{dy}{dx}$ is equal to:

(a) $\frac{1}{x}$ (b) $\frac{1}{x \ln a}$ 12302175

(c) $\frac{\ln a}{x}$ (d) $x \ln a$

50. $\frac{d}{dx}\left(\log_{10} x\right) =$ _____ 12302176

(a) $\frac{1}{x}$ (b) $\frac{1}{x \ln 10}$

(c) $x \ln 10$ (d) $x \cdot \frac{1}{\sqrt{x^{10}}}$

51. $\frac{d}{dx}\left(\ln \sqrt{x}\right)$ **is equal to:** 12302177

(a) x (b) $\frac{1}{x}$

(c) $\frac{1}{\sqrt{x}}$ (d) None of these

Equation of Tangent and Normal Line to a Curve

52. The tangent line to the curve $y = f(x)$ at the point $(x_0, f(x_0))$ has slope:

(a) $f(x_0)$ (b) $f'(x_0)$ 12302178

(c) $f''(x_0)$ (d) $\frac{1}{f'(x_0)}$

53. The equation of tangent at point (x_0, y_0) is: 12302179

(a) $y - y_0 = m(x - x_0)$

(b) $y + y_0 = m(x + x_0)$

(c) $y = mx$

(d) $y - y_0 = \frac{-1}{m}(x - x_0)$

54. The tangent line exists provided: 12302180

(a) $f(x_0) = 0$ (b) $f'(x_0)$ exists

(c) $x_\circ = 0$ (d) $n = 0$

55. A normal line is a line that is: 12302181

(a) Parallel to tangent

(b) Perpendicular to tangent

(c) Parallel to x-axis

(d) Parallel to y-axis

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56. The equation of normal line at point (x_0, y_0) is: 12302182

(a) $y - y_0 = m(x - x_0)$

(b) $y - y_0 = \frac{-1}{m}(x - x_0)$

(c) $y + y_0 = m(x - x_0)$

(d) $y = mx$

57. If the slope of tangent line is 'm' then the slope of normal line is: 12302183

(a) m (b) $\frac{1}{m}$

(c) $\frac{-1}{m}$ (d) $-m$

58. If $f'(x_0) = 0$, then equation of the normal lien is: 12302184

(a) $y - y_0$ (b) $x = x_0$

(c) $y = x$ (d) $x + y = 0$

59. A vertical line occurs when the slope is:

(a) 0 (b) 1 12302185

(c) ∞ (d) -1

60. The slope of tangent line is also called the: 12302186

(a) Curvature

(b) Slope of the curve

(c) Radius of the curve

(d) None of these

Higher-Order Derivatives

61. The derivative of a derivative is called a: 12302187

(a) Function

(b) Higher-order derivative

(c) Constant

(d) Variable

62. The second order derivative of a $f(x)$ is denoted by: 12302188

(a) $f'(x)$ (b) $f''(x)$

(c) $f'''(x)$ (d) $f^n(x)$

63. The third order derivative of $f(x)$ is denoted by: 12302189

(a) $f(x)$ (b) $f'(x)$

(c) $f''(x)$ (d) $f'''(x)$

64. The n^{th} ordered derivative of $f(x)$ is denoted by: 12302190

(a) $f'(x)$ (b) $f''(x)$

(c) $f'''(x)$ (d) $f^n(x)$

65. Higher derivatives are obtained by:

(a) Integration 12302191

(b) Repeated differentiation

(c) Factorization

(d) Simplification

66. The n^{th} derivatives of 'y' is denoted by: 12302192

(a) $y^{(n)}$ (b) y'

(c) y'' (d) y'''

67. If $y = x^3 + 2x$ then y_2 is: 12302193

(a) $3x^2 + 2$ (b) $6x$

(c) $2x^3 + 2$ (d) $2x + 2$

Increasing and Decreasing Function

68. A function 'f' is called increasing on an interval I if: 12302194

(a) $f(x_1) \geq f(x_2)$, whenever $x_1 < x_2$

(b) $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$

(c) $f(x_1) = f(x_2)$ for all x_1, x_2

(d) None of these

69. A function is called decreasing on an interval I if: 12302195

(a) $f(x_1) \leq f(x_2)$, whenever $x_1 < x_2$

(b) $f(x_1) \geq f(x_2)$, whenever $x_1 < x_2$

(c) $f(x_1) = f(x_2)$

(d) $f'(x) > 0$

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70. Which interval represent all real numbers: 12302196
(a) $(0,\infty)$ (b) $(-\infty,\infty)$
(c) $(-\infty,0)$ (d) $[0,\infty)$

71. If $f'(x)\geq 0$, for every 'x' in an interval I, then f is: 12302197
(a) Constant (b) Increasing
(c) Decreasing (d) Undefined

72. If $f'(x)\leq 0$, for every 'x' in an interval I, then f is: 12302198
(a) Increases (b) Constant
(c) Decreasing (d) Undefined

73. The interval (a,b) means both '*a*' and '*b*' are: 12302199
(a) Included (b) Excluded
(c) 'a' include (d) only 'b' included

74. In interval $[a,b]$ means both '*a*' and '*b*' are: 12302200
(a) Included
(b) Excluded
(c) Only 'a' excluded
(d) Only 'b' excluded

75. A point on the graph where its derivative is zero is called…. 12302201
(a) Stationary point
(b) Increasing point
(c) Decreasing point
(d) Vertex

76. Every stationary point is a ____ point.
(a) Critical (b) Vertex 12302202
(c) Increasing (d) Decreasing

77. In each sub-interval of the domain determined by the critical points, the sign of the derivative remains _____. 12302203
(a) Same (b) Change
(c) Positive (d) Negative

78. $f(x)$ has critical pint where: 12302204
(a) $f(x)=0$
(b) $f'(x)=0$ or $f'(x)$ does not exist
(c) $f'(x)=1$
(d) $f'(x)$ is maximum only

79. The function $f(x)=|x|$ has a critical point at: 12302205
(a) $x=1$ (b) $x=-1$
(c) $x=0$ (d) No, critical point

80. The function $f(x)=x^2$ has critical point at: 12302206
(a) $x=0$ (b) $x=1$
(c) $x=-1$ (d) $x=2$

81. The critical point of the function $f(x)=2x^3-3x^2-12x$ is: 12302207
(a) $x=-1,-2$ (b) $x=-1,2$
(c) $x=1,2$ (d) $x=1,-2$

Relative and Absolute Exterma

82. A function f has a global (absolute) maximum at $x=c$ if: 12302208
(a) $f(c)\leq f(x)$ for all 'x'
(b) $f(c)\geq f(x)$, for all x
(c) $f'(c)=0$
(d) $f(c)=0$

83. If $f(x)$ has a global (absolute) maximum at $x=c$, then $f(c)$ is called global (absolute)_____ value of the function. 12302209
(a) Minimum (b) Maximum
(c) Constant (d) Undefined

84. A function has a global (absolute) minimum at $x=c$ if: 12302210
(a) $f(c)\geq f(x)$, for all x
(b) $f(c)\leq f(x)$, for all x
(c) $f'(c)>0$
(d) $f(c)$ is undefined

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85. If $f(x)$ has global (absolute) minimum at $x=c$, then $f(c)$ is called global (absolute)_____ value of the function. 12302211

(a) Minimum (b) Maximum
(c) Constant (d) Undefined

86. The global maximum and minimum of a function are collectively called:

(a) Stationary point 12302212
(b) Critical point
(c) Global externa
(d) Turning points

87. For $f(x)=x^2$ on the interval $[0,2]$ the global maximum occur at: 12302213

(a) $x=0$ (b) $x=1$
(c) $x=2$ (d) $x=1.5$

88. For function $f(x)=x^2$ on the interval $[0,2]$ the global minimum value occurs at: 12302214

(a) $x=0$ (b) $x=1$
(c) $x=2$ (d) $x=1.5$

89. If first derivative test is not defined at a critical point, we can apply__ derivative test at that critical point.

(a) First (b) Second 12302215
(c) Third (d) Fourth

90. According to second derivative test, f has relative maximum at c if: 12302216

(a) $f''(c)<0$ (b) $f''(c)>0$
(c) $f''(c)=0$ (d) $f''(c)=\infty$

91. According to second derivative test if $f''(c)=0$ then second derivative test is: 12302217

(a) Valid (b) Inclusive
(c) Undefined (d) All of these

Local Linear Approximation (Linearization)

92. If $f(x)$ is differentiable at $x=a$, then the linearization of $f(x)$ at $x=a$ is:

(a) $L(x)=f(a)$ 12302218
(b) $L(x)=f(a)+f'(a)(x-a)$
(c) $L(x)=f(a)-f'(a)(x-a)$
(d) $f(x)=f(x)+a$

93. The approximation of $f(x)$ by $L(x)$ is called the …. approximation. 12302219

(a) Global (b) Local linear
(c) Quadratic (d) Polynomial

94. The point $x=a$ in linearization is referred to as the_____. 12302220

(a) Critical point
(b) Training point
(c) Center of the approximation
(d) End point

95. Linear function has form: 12302221

(a) $y=mx+c$ (b) $y=mx^2+c$
(c) $y=c$ (d) $y=mx^3+x+c$

96. The graph of linear function is _____ line. 12302222

(a) Straight (b) Curve
(c) Zigzag (d) V-Shape

Differentials

97. The differential dy is defined by: 12302223

(a) $dy=f(x)dx$ (b) $dy=f'(x)dx$
(c) $dy=f''(x)dx$ (d) $dy=xdx$

98. The variable treated independent in the definition of differential $dy=f'(x)dx$ is: 12302224

(a) x (b) y
(c) dy (d) dx

99. The relation obtained after dividing $dy=f'(x)dx$ by dx is: 12302225

(a) $\frac{dx}{dy}=f'(x)$ (b) $\frac{dy}{dx}=f'(x)$
(c) $\frac{dy}{dx}=f(x)$ (d) $\frac{dx}{dy}=f(x)$

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100. When a function $y = f(x)$ is differentiable at a point $P(x, y)$. Then actual change Δy in y is: 12302226

(a) $f'(x)dx$ (b) $f(x+\Delta x)-f(x)$

(c) $f(x)-f'(x)$ (d) $f(x+\Delta x)+f(x)$

101. When $dx = \Delta x$, the differential is written as: 12302227

(a) $dy = f'(x)\Delta x$ (b) $dy = f(x)\Delta x$

(c) $dy = \Delta y$ (d) $dy = \dfrac{\Delta y}{\Delta x}$

Error In Differential Approximation

102. The approximation change in 'y' obtained using differentials is called:

(a) Δy (b) dy 12302228

(c) dx (d) $f(x)$

103. The error in Δy is defined as: 12302229

(a) $dy-\Delta y$ (b) $\Delta y-dy$

(c) $\Delta y+dy$ (d) $\Delta y\times dy$

104. Percentage error is obtained by multiplying the relative error by:

(a) 10 (b) 50 12302230

(c) 100 (d) 1000

105. The formula for relative error and percentage error are valid only when:

(a) $dx = 0$ (b) $\Delta y = 0$ 12302231

(c) $\Delta y \neq 0$ (d) $f'(x) = 0$

106. Percentage error is equal to: 12302232

(a) $\left|\dfrac{\Delta y - dy}{\Delta y}\right|\times 100$ (b) $\left|\dfrac{\Delta y + dy}{\Delta x}\right|\times 100$

(c) $\left|\dfrac{\Delta y + \Delta x}{dx}\right|\times 100$ (d) $\left|\dfrac{\Delta x + dx}{\Delta y}\right|\times 100$

Applying Exterma to Real World Problems

107. The first step in solving extrema problems is to: 12302233

(a) Find critical points

(b) Understand the problem

(c) Verify extrema

(d) Formulate the function

108. The second derivative test is used to confirm: 12302234

(a) Domain and range

(b) Maxima and minima

(c) Continuity

(d) Differentiability

ANSWER KEY

1.	b	19.	b	37.	c	55.	b	73.	b	91.	b
2.	c	20.	b	38.	b	56.	b	74.	a	92.	b
3.	d	21.	b	39.	c	57.	c	75.	a	93.	b
4.	d	22.	c	40.	b	58.	b	76.	a	94.	c
5.	c	23.	c	41.	a	59.	c	77.	a	95.	a
6.	b	24.	d	42.	c	60.	b	78.	b	96.	a
7.	c	25.	c	43.	b	61.	b	79.	c	97.	b
8.	b	26.	c	44.	a	62.	b	80.	a	98.	d
9.	b	27.	a	45.	b	63.	d	81.	b	99.	b
10.	c	28.	b	46.	a	64.	d	82.	b	100.	b
11.	b	29.	a	47.	b	65.	b	83.	b	101.	a
12.	b	30.	c	48.	c	66.	a	84.	b	102.	b
13.	b	31.	a	49.	b	67.	b	85.	a	103.	b
14.	b	32.	b	50.	b	68.	b	86.	c	104.	c
15.	c	33.	d	51.	d	69.	b	87.	c	105.	c
16.	a	34	b	52.	b	70.	b	88.	a	106.	a
17.	b	35.	c	53.	a	71.	b	89.	b	107.	b
18.	c	36.	a	54.	b	72.	c	90.	a	108.	b